# UNIVERSAL TYPE SPACES WITH UNAWARENESS

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Abstract. Hierarchies of awareness and belief arise in games with unawareness, similarly to belief hierarchies in standard games. A natural question is whether a model generating awareness and belief hierarchies closes, i.e. whether each hierarchy describes the player's awareness of the hierarchies of other players and beliefs over these. This paper proves that the model closes by constructing universal type spaces for belief and unawareness, where each type has a belief over all types. An alternative proof is possible if topology is added to the type spaces. Unawareness of agents and unawareness of higher-order reasoning permit a similar construction of a universal type space as propositional unawareness.

## 1. Introduction

There are many situations where the payoff of an agent depends on the actions of other agents and on uncertain external factors. The actions of the agents depend on their reasoning, so to choose the best action, an agent has to reason about the reasoning of others, their reasoning about the reasoning of others, and so on to arbitrarily high order. With complete information, this infinite regress can be avoided by imposing a fixed-point equilibrium concept, but with incomplete information, the process generates infinite hierarchies of reasoning. A natural question is whether the infinite hierarchies summarize all the uncertainty in the game or whether it is necessary to go even further, describing a player's reasoning about the opponents' infinite hierarchies, about their reasoning about their opponents' hierarchies, etc.

In standard games, the reasoning of an agent is described by a probability distribution over known outcomes (exogenous uncertainties and other agents' possible beliefs). The seminal paper of Mertens and Zamir (1985) was the first to show that the initial hierarchies of reasoning are sufficient—each hierarchy encodes a probability distribution over the set of hierarchies, thus closing the model. This result has been extended to more general spaces by Brandenburger and Dekel (1993). The assumptions are relaxed further in Heifetz and Samet (1998b), but at the cost of not including all hierarchies.

Not all uncertainty is describable by a probability distribution, but infinite regress is a feature of any kind of interactive reasoning. The same question of closure of the hierachical model then arises as for probabilistic beliefs. If the uncertainty is described by conditional probability systems, compact continuous possibility correspondences or compact sets of probabilities, the papers of Battigalli and Siniscalchi (1999); Mariotti, Meier, and Piccione (2005) and Ahn (2007) show that the hierarchies capture all uncertainty about the hierarchies.

On the other hand, for knowledge and similar information structures, reasoning about other agents' reasoning may continue indefinitely, as shown in Heifetz and Samet (1998a); Meier (2005), therefore there may be no level of the hierarchies that fully describes the uncertainty in the model. With additional assumptions on the knowledge operators, closure can be obtained (Meier, 2008; Mariotti, Meier, and Piccione, 2005).

All the abovementioned kinds of uncertainty share the property that the agents know all possible outcomes. In many real-world decision problems this is not the case—the agents may be unaware of some aspects of the environment. This may give rise to novel behaviour, as illustrated in the game in Fig. 1. Choice under unawareness may exhibit 'reverse Bayesianism', i.e. updating may lead an initially null event to receive positive weight in decisions (Karni and Vierø, 2010). The experiment of Mengel, Tsakas, and Vostroknutov (2011) shows that being exposed to unawareness increases the risk aversion of experimental subjects.

To better describe such situations, unawareness has been added to games by Feinberg (2009); Grant and Quiggin (2009); Halpern (2008); Rêgo and Halpern (2007); Heifetz, Meier, and Schipper (2011a). In games with unawareness, the players are aware of only some aspects of the game, form beliefs about external uncertainties and other players' awareness and beliefs, and so on, giving rise to infinite hierarchies of awareness and belief. Similarly to standard games, there is a question as to whether the hierarchies encapsulate all uncertainty in the model. Given the results

*Date*: November 25, 2011.
The author is grateful to Larry Moss, Aviad Heifetz, Eduardo Faingold, Larry Samuelson and Miklós Pintér for helpful comments and suggestions. The author thanks participants of the conference Unawareness: Conceptualization and Modeling at Johns Hopkins and seminar participants at Yale and Stanford GSB.

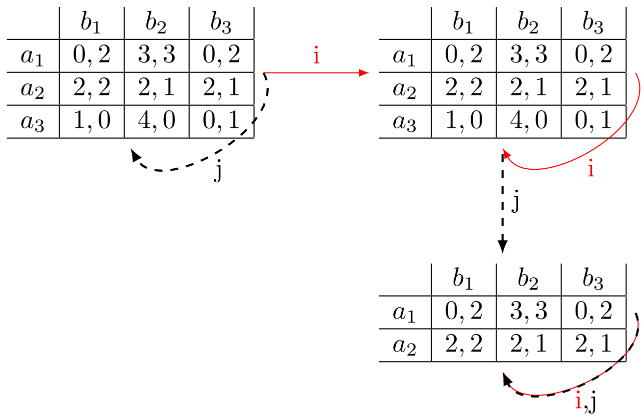



FIGURE 1. A game with unawareness from Feinberg (2009). Beliefs are represented by arrows, solid for player $i$ and dashed for player $j$. At the top left game $j$ believes the game to be the top left, while $i$ believes it to be the top right. The solution is $(a_3, b_3)$, while without unawareness, it would be $(a_2, b_1)$ in the top games and (in pure strategies) either $(a_2, b_1)$ or $(a_1, b_2)$ in the bottom game (Feinberg, 2009).

in the literature that the hierarchical model closes for some kinds of uncertainty, but not for others, the answer is not obvious in the case of unawareness.

This paper proves that in the case of propositional unawareness, unawareness of agents, unawareness of higher-order reasoning, or a combination of these, the infinite hierarchies of awareness and belief include the description of awareness and belief about the hierarchies. Modelling agent-based unawareness and combined kinds of unawareness together with probabilistic beliefs appears to be new. Together with knowledge, many kinds of unawareness have been studied in a propositional model using modal logic by Fagin and Halpern (1988).

In propositional unawareness, the awareness levels are based on partitions of the states of nature. An agent with a particular awareness level can reason about the partition corresponding to that level, about others' reasoning about that partition, their reasoning about his reasoning about the partition, etc. Propositional unawareness is the most common kind used in the literature, in both propositional models (Halpern, 2001; Heifetz, Meier, and Schipper, 2008; Halpern and Rêgo, 2008) and set-based ones (Modica and Rustichini, 1999; Heifetz, Meier, and Schipper, 2006; Li, 2009).

Under unawareness of agents, all agents have the same partition of the states of nature. Awareness levels are based on sets of agents, so an agent with limited awareness is able to reason only about a subset of agents, their reasoning about this subset of agents, their reasoning about his reasoning about this subset, etc. This kind of unawareness may be especially relevant for games with unawareness.

Unawareness of higher-order reasoning is an extension of Kets (2010) to purely measurable spaces. It seems the least natural of the three kinds of unawareness considered here. Bounded reasoning may be the result of other factors besides unawareness, e.g. computational limitations, time constraints or heuristic decision making. One instance where unawareness is a natural cause of bounded reasoning is when decision makers lack a theory of mind, such as animals or young children. In that case the agents are intrinsically unable to reason about the beliefs of others.

A combination of the above kinds of unawareness means that the agent can only reason about a coarse partition of the states of nature, a subset of the other agents reasoning about that partition, their reasoning about the reasoning of that subset of agents and so on, up to a finite order.

Not all kinds of unawareness are considered in this paper. Unawareness of a particular belief (e.g. someone putting probability $\frac{1}{3}$ on it snowing is inconceivable to an agent) is ruled out, as is unawareness of lower orders of belief while being aware of higher orders. The agent can put zero probability on these beliefs, but not be unaware of them. Syntactic unawareness, under which the agent can be aware of 'it rains or it snows' but unaware of 'it snows or it rains', is also excluded.

Two other papers independently proving the existence of the universal type space with propositional unawareness are Heifetz, Meier, and Schipper (2011b) and Pintér and Udvari (2011). Both study propositional unawareness, and Pintér and Udvari also cover bounded reasoning along the lines of Kets (2010). In the present paper and in that of Heifetz, Meier, and Schipper, the awareness levels are sets of types whose beliefs have the same level of detail, while in Pintér and Udvari (2011) the awareness levels are $\sigma$-algebras on a common space. Due to the different definition,

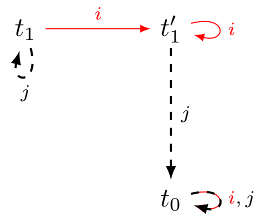



FIGURE 2. A type space with unawareness. Beliefs are represented by arrows, solid for player $i$ and dashed for player $j$. At type $t_1$, agent $j$ is aware of $t'_1$, but puts probability zero on it. At type $t'_1$, agent $j$ is unaware of $t'_1$ and $t_1$.

unawareness in Pintér and Udvari's paper is subject to the Dekel, Lipman, and Rustichini (1998) impossibility result discussed below and should be interpreted differently from other unawareness models.

In Heifetz, Meier, and Schipper's paper, the proof of universality requires topological assumptions, which is not the case in the present paper and that of Pintér and Udvari. The present paper also describes unawareness of agents and the combination of different kinds of unawareness, which the other two papers do not.

Unawareness type spaces have been used to prove no-trade and agreement theorems (Heifetz, Meier, and Schipper, 2007). Other potential applications of universal type spaces with unawareness mirror those of universal belief type spaces—studying common priors (Mertens and Zamir, 1985), common certainty of the structure (Brandenburger and Dekel, 1993) and robustness of solution concepts to perturbations of information (Chen, Di Tillio, Faingold, and Xiong, 2010). Type spaces with unawareness enable games with unawareness to be analyzed, in addition to standard games.

The structure of the universal type space with unawareness constructed here reflects the structure of games with unawareness. Games with unawareness consist of a tree of standard games (as in Fig. 1), partially ordered by inclusion of the set of nodes (Feinberg, 2009; Heifetz, Meier, and Schipper, 2011a). Each game describes the view of an agent at a particular awareness level. In each node of each game, the players perceive themselves to be in some node in one of the weakly smaller games, e.g. in Fig. 1 at the top right game, agent $j$ perceives himself to be in the smaller game at the bottom. The awareness level of a player at a node is determined by the game the player perceives himself to be in, not the game he is actually in. An agent attributes to others weakly lower awareness than his own level, e.g. in Fig. 1 in the top left game, $j$ attributes the same awareness to $i$ as to himself, while $i$ attributes lower awareness to $j$.

In the type spaces with unawareness in this paper, such as the one in Fig. 2, each type encodes the state of nature and the belief of each player. Each type is labelled with an awareness level (the levels are 0 and 1 in Fig. 2). The belief of a player in a type is over types with weakly less awareness, which reflects believing weakly smaller games. The level of types that an agent believes is that agent's awareness level, which mirrors the situation in games with unawareness where the believed game is the awareness level of a player. Uncertainty about the beliefs and awareness of others is captured by a probability distribution over their types, which in games corresponds to a belief over the nodes in a player's information set.

Unawareness is different from probability zero in games with unawareness and the corresponding type spaces, e.g. unawareness is symmetric—if an agent is unaware of an event, he is unaware of its negation, while putting probability zero on an event implies putting probability one on its negation. This is illustrated in Fig. 2, where at type $t'_1$, agent $j$ is unaware of both $t_1$ and $t'_1$, while at type $t_1$, agent $j$ puts probability one on $t_1$ and zero on $t'_1$.

The universal type space with unawareness in this paper, like any model with unawareness, has to bypass the impossibility result of Dekel, Lipman, and Rustichini (1998) that standard state spaces preclude unawareness. Given natural properties of knowledge and unawareness, the result of Dekel, Lipman, and Rustichini rests on two basic axioms. The first axiom (real states) requires the negation and conjunction operators to behave in the standard way, and the second (event sufficiency) requires the operators for knowledge and awareness to take events to events. Knowledge and awareness of a statement cannot depend on other factors, such as the syntactic form of the statement. As pointed out by Heifetz, Meier, and Schipper (2006); Schipper (2011), one of these axioms must be relaxed to model nontrivial unawareness. Economics models usually relax real states, while the logic literature mostly forgoes event sufficiency.

This paper follows the economics literature in omitting the real states axiom and keeping event sufficiency. This means negation and disjunction are nonstandard, but awareness depends only on the set of states belonging to an event, not on how the event is constructed or expressed. Section 4 shows that the desirable properties of knowledge and awareness listed in Dekel, Lipman, and Rustichini (1998) hold in the model of this paper, and section 3 shows awareness is nontrivial. Therefore the universal type space with unawareness bypasses the impossibility result.

The next section proves the existence of the universal type space with propositional unawareness and shows how it may be unpacked into hierarchies of belief and awareness. Section 3 describes the events in the model and the belief and awareness operators. The connection to Dekel, Lipman, and Rustichini (1998) is described in section 4. Unawareness of agents, of higher order reasoning, and a combination of three kinds of unawareness are considered in section 5. The construction with a topology is presented in section 6 and compared to the model of Heifetz, Meier, and Schipper (2011b).

## 2. The existence of universal type spaces with unawareness

In this section, propositional unawareness is added to the universal space of belief hierarchies of Heifetz and Samet (1998b). Theirs is chosen as the basic framework, because unlike the earlier belief hierarchies of Mertens and Zamir (1985); Brandenburger and Dekel (1993), it has no topological structure and only uses measurability assumptions. A topology of the type space is not a necessary or natural component of descriptions of beliefs, as argued in Heifetz and Samet (1998b); Pintér (2010), but in the topological case, an alternative proof is possible.

The existence proof of the universal type space relies on the category-theoretic results of (Moss and Viglizzo, 2006, Theorem 6.4) and (Viglizzo, 2005, Theorem 7.1), which use purely measurable spaces. When topological structure is added to make all spaces Polish, the alternate universality proof (Theorem 11 below) uses the results of Schubert (2008); Doberkat and Schubert (2011).

### 2.1. Notation.

Fix for most of the paper a measurable space $S$ as the set of states of nature and a finite set $I$ as the set of agents (denoted $i, j \in I$). Before defining the universal type space and proving its existence, some notation is needed. Let $\bigsqcup$ denote disjoint union. The complement of $E$ is $E^c$. Nature is treated notationally as agent 0 and $I_0 = I \cup \{0\}$.

For a measurable space $M$, its $\sigma$-algebra and the coarsenings of that $\sigma$-algebra are denoted by $\mathcal{G}_M$, $\mathcal{F}_M$, $\mathcal{E}_M$, $\mathcal{D}_M$, with $\mathcal{G}_M$ being the finest. The subscript is omitted when no confusion should arise. The coarsest common refinement of a set of $\sigma$-algebras $\{\mathcal{F}_n : n \in N\}$ is denoted $\vee_n \mathcal{F}^n$. The notation $\mathcal{F} \triangleright \mathcal{E}$ means that $\mathcal{F}$ is a refinement of $\mathcal{E}$.

For a measurable space $M$ with $\sigma$-algebra $\mathcal{G}$, denote by $\Delta(M)$ the set of probability measures over $X$, endowed with the $\sigma$-algebra generated by

$$\{\beta^q(E) : E \in \mathcal{G},\ q \in [0,1]\},\ \text{where } \beta^q(E) = \{\mu \in \Delta(M) : \mu(E) \geq q\}.$$

Abusing notation, the $\sigma$-algebra on $\Delta(M)$ generated this way from $\mathcal{G}$ is also denoted $\mathcal{G}$, and similarly for $\Delta(\Delta(M))$ etc. As usual, $\delta_t$ denotes the Dirac delta function on $t \in M$, i.e. the probability distribution concentrated on $t$.

The product of a vector of measurable spaces $(M_i)_{i \in I_0}$ is $\times_{i \in I_0} M_i = M$ and the product of all spaces except $i$ in the vector is $\times_{j \in I_0 \setminus \{i\}} M_i = M_{-i}$. A similar convention applies to collections of measurable spaces with more than one index $(M_{\mathcal{F},i})^{i \in I_0}_{\mathcal{F} \triangleleft \mathcal{G}}$, and to collections of measurable functions. The product over one index keeps the other index, $M_i = \times_{\mathcal{F} \triangleleft \mathcal{G}} M_{\mathcal{F},i}$, and the product over both indexes drops both, $M = \times_{\mathcal{F} \triangleleft \mathcal{G},\, i \in I_0} M_{\mathcal{F},i}$. Abusing notation, the collection $(M_{\mathcal{F},i})^{i \in I_0}_{\mathcal{F} \triangleleft \mathcal{G}}$ may also be denoted $M$. Products of measurable spaces have the product $\sigma$-algebra.

With propositional unawareness, the set of awareness levels is generated by a complete semilattice of $\sigma$-algebras on the states of nature, closed under taking coarsest common refinements. The whole lattice is $\{\mathcal{F} : \mathcal{F} \triangleleft \mathcal{G}\}$, where $\mathcal{G}$ is the original $\sigma$-algebra on $S$. One example of such a semilattice can be constructed by starting from a collection of measurable sets $\{E_n \subseteq S : n \in N\}$, taking for each $E_n$ the $\sigma$-algebra $\mathcal{F}^n = \{E_n, E_n^c, S, \emptyset\}$, and adding all coarsest common refinements of sets of $\mathcal{F}^n$. These $\sigma$-algebras on states of nature are extended to the whole type space by allowing agents with awareness $\mathcal{F}$ to distinguish only events in $\mathcal{F}$, others' beliefs differing on these events, their second order beliefs differing on beliefs differing on these events, etc. For each $\mathcal{F}$, the set $S$ with the $\sigma$-algebra $\mathcal{F}$ is denoted $S_{\mathcal{F}}$.

### 2.2. Definitions and existence.

The definitions of type spaces with unawareness, type morphisms and the universal type space with unawareness are presented next, followed by the existence proof. Type spaces with belief and propositional unawareness are defined the same way here as in Heifetz, Meier, and Schipper (2007)[1], since both papers extend the construction of Heifetz, Meier, and Schipper (2006) to the probabilistic case.

**Definition 1.** For states of nature $S$, agents $I$ and maximal awareness level $\mathcal{G}$, a type space with propositional unawareness is $(M, g) = ((M_{\mathcal{F},i})^{i \in I_0}_{\mathcal{F} \triangleleft \mathcal{G}}, (g_{\mathcal{F},i})^{i \in I_0}_{\mathcal{F} \triangleleft \mathcal{G}})$ such that

(i) $M_{\mathcal{F},i}$ is a measurable space for each $\mathcal{F}$ and $i$, and $M_{\mathcal{F},0} = S_{\mathcal{F}}$ for each $\mathcal{F}$,

[1] The author thanks Aviad Heifetz for pointing out this connection.

(ii) for each $\mathcal{F}$ and $i \in I$, the type function $g_{\mathcal{F},i} : M_{\mathcal{F},i} \to \bigsqcup_{\mathcal{E} \lhd \mathcal{F}} \Delta(M_{\mathcal{E},-i})$ is measurable,
(iii) for each $\mathcal{F}$, the state of nature function $g_{\mathcal{F},0} : M_{\mathcal{F},0} \to S_{\mathcal{F}}$ is the identity.

The type function $g_i$ for agent $i$ consists of all the $g_{\mathcal{F},i}$. For ease of notation, $g_i$ may be written as mapping $M$ to $\Delta(M_{-i})$, with the understanding that $g_i$ only depends on the $i$-th coordinate of its argument. The full vector $g$ maps $M$ to the vector of measurable spaces

$$V(M) = \left( S_{\mathcal{F}}, \left( \bigsqcup_{\mathcal{E} \lhd \mathcal{F}} \Delta(M_{\mathcal{E},-i}) \right)^{i \in I} \right)_{\mathcal{F} \subseteq \mathcal{G}} \tag{1}$$

For type $t \in M_{\mathcal{F}}$ for some $\mathcal{F}$, the beliefs of $i$ are over $M_{\mathcal{E},-i}$ for some $\mathcal{E} \lhd \mathcal{F}$, so compared to the space where $t$ is drawn from, the dimension $M_{\mathcal{E},i}$ is missing. To obtain higher order belief, $g_i$ must be iterated, so its domain is implicitly extended to the missing dimension in a way that ensures the agent is certain of his beliefs modulo awareness, $g_i(t) \in \Delta(M_{\mathcal{E},-i}) \times \delta_{t_{\mathcal{E},i}}$. The type $t_{\mathcal{E},i}$ that $i$ is certain of at $t_i$ is the natural projection of $t_i$ to $t_i$'s awareness level $\mathcal{E}$, meaning $g_i(t_i) = g_i(t_{\mathcal{E},i})$. If different types have different beliefs, there is a unique such $t_{\mathcal{E},i}$. If many $t_{\mathcal{E},i}$ satisfy the condition, we can pick any of them as the extension. There is no circularity in the definition of the extension, since $t_{\mathcal{E},i}$ is chosen based on the equality of $g_i(t_i)$ and $g_i(t_{\mathcal{E},i})$ as elements of $\Delta(M_{\mathcal{E},-i})$ before the extension, not as elements of $\Delta(M_{\mathcal{E}})$ after the extension.

For type spaces $M'$ and $M$ with $V(M)$, $V(M')$ defined by Eq.~(1) and for a vector of measurable functions $z = (z_{\mathcal{F},i})^{i \in I_0}_{\mathcal{F} \lhd \mathcal{G}} : M' \to M$ mapping one type space to another, define the function $V(z) : V(M') \to V(M)$ mapping the beliefs in one type space to the beliefs in another as follows. First set $V(z_{\mathcal{F},0}) = z_{\mathcal{F},0}$ for all $\mathcal{F} \lhd \mathcal{G}$. For $i \in I$, $\mathcal{F} \lhd \mathcal{G}$ and $\mu'_{\mathcal{F},i} \in \Delta(M'_{\mathcal{F},-i})$, let

$$V(z)(\mu'_{\mathcal{F},i}) = \mu_{\mathcal{F},i} \in \Delta(M_{\mathcal{F},-i}) \text{ s.t. } \mu_{\mathcal{F},i}(E) = \mu'_{\mathcal{F},i}(z^{-1}_{\mathcal{F},-i}(E)) \quad \forall E \subseteq M_{\mathcal{F},-i} \text{ measurable.} \tag{2}$$

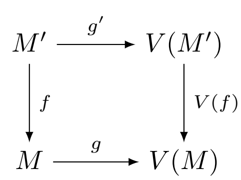
To connect type spaces with unawareness to each other, type morphisms are needed. These are maps from one type space to another that preserve belief and awareness information.

**Definition 2.** A type morphism between $(M', g')$ and $(M, g)$ is a vector $f = (f_{\mathcal{F},i})^{i \in I_0}_{\mathcal{F} \lhd \mathcal{G}}$ of measurable functions $f_{\mathcal{F},i} : M'_{\mathcal{F},i} \to M_{\mathcal{F},i}$, such that

$$g_i(f(t'))(E) = g'_i(t')(f^{-1}_{\mathcal{D},-i}(E)) \quad \forall i \in I \quad \forall \mathcal{D} \lhd \mathcal{G} \quad \forall t' \text{ s.t. } g'_i(t') \in \Delta(M'_{\mathcal{D},-i}) \quad \forall E \subseteq M_{\mathcal{D},-i} \text{ measurable.} \tag{3}$$

A type morphism $f$ is an isomorphism if it is a measure-preserving bijection between $M'$ and $M$, i.e. $f^{-1} : M \to M'$ is also a type morphism.

The intuition for Eq. (3) that a type morphism must satisfy is illustrated in the following commutative diagram, where $V(M)$ is defined in Eq. (1) and $V(f)$ in Eq. (2). Finding first the beliefs of a type in the original space and then transferring them to the other space has to give the same result as first transferring the type to the other space and then finding its beliefs there, i.e. $g \circ f = V(f) \circ g'$.




The definition of a universal type space is standard—it is a type space containing all other type spaces.

**Definition 3.** A type space with unawareness $(\Omega, g)$ is universal if for every type space $(M', g')$ with the same set of agents $I$, awareness levels $\{\mathcal{F}^n : n \in N\}$ and states of nature $S$ there is a unique type morphism from $(M', g')$ to $(\Omega, g)$.

The existence and uniqueness of a universal type space for propositional unawareness is proved in Theorem 1. The proof in the appendix translates the problem into category theory and uses Viglizzo (2005), which generalizes Heifetz and Samet (1998b). The proof is similar to Pintér and Udvari (2011), but the definition of a type space differs from that paper.

The universal type space in the present section is the set of all hierarchies of belief and awareness that some type in some type space maps to. The set of all hierarchies is constructed in subsection 2.3. If the states of nature form a Polish space, the universal type space is the set of all consistent hierarchies of belief and awareness. The proof of this is a special case of Theorem 3 in Schubert (2008) and is given in subsection 6. For propositional unawareness, Heifetz, Meier, and Schipper (2011a) independently prove the existence of a universal type space under the assumption that the states of nature form a Hausdorff topological space. The proof in this section differs from Schubert (2008); Heifetz, Meier, and Schipper (2011a) because it does not use any topological assumptions.

**Theorem 1.** *For fixed sets of states of nature $S$, agents $I$ and awareness levels $\{\mathcal{F}^n : n \in N\}$, there exists a universal type space with propositional unawareness $(\Omega, g)$ unique up to isomorphism. In $(\Omega, g)$, the type function $g$ is an isomorphism.*

For all type spaces with the sets of states of nature, agents or awareness levels being $S' \subseteq S$, $I' \subseteq I$ and $\{\mathcal{F}^n : n \in N' \subseteq N\}$ respectively, the universal type space for $S$, $I$ and $\{\mathcal{F}^n : n \in N\}$ is weak-universal, meaning a type morphism into it exists, but is not unique. In particular, a universal type space for $S'$, $I'$ and $\{\mathcal{F}^n : n \in N' \subseteq N\}$ can be embedded in the universal space for $S$, $I$ and $\{\mathcal{F}^n : n \in N\}$. The embedding is nonunique if labels of states, agents or awareness levels can be changed, e.g. agent 1 in a type space with $I = \{1\}$ can be mapped to either agent 1 or 2 in a type space with $I = \{1, 2\}$. If labels must be preserved, the embedding is unique, but may not exist when intuitively it should: a type space with $I = \{1, 2\}$ cannot then be embedded in a space with $I = \{2, 3\}$ or vice versa.

2.3. **From types to belief hierarchies with unawareness.** Two proofs of the existence of the universal space are given in Viglizzo (2005)—one constructs the space so that a type equals the set of all events which contain the type, the other shows that a subset of the set of all hierarchies is the universal type space. Section 2.2 showed that the universal type space with unawareness exists, while section 3 will describe the events in the universal space and connect a type to the events containing it. The present subsection unpacks types into hierarchies of belief and awareness.

First the set of all hierarchies is defined and then the function from types to hierarchies is given. The construction is similar to Heifetz and Samet (1998b). The base case of the inductive construction of hierarchies of belief and awareness is setting $H^0_{\mathcal{F},i}$ to be a singleton for all $i \in I$ and $\mathcal{F} \lhd \mathcal{G}$, and setting $H^k_{\mathcal{F},0} = S_{\mathcal{F}}$ for all $k \geq 0$ and all $\mathcal{F} \lhd \mathcal{G}$.

The inductive step is to apply the transformation $V$ from Eq. (1) an infinite number of times to $\left(H^0_{\mathcal{F},i}\right)^{i \in I_0}_{\mathcal{F} \lhd \mathcal{G}}$. Therefore $H^{k+1} = V(H^k)$, which in more detail is $H^{k+1}_{\mathcal{F},0} = S_{\mathcal{F}}$ and for $i \neq 0$,

$$H^{k+1}_{\mathcal{F},i} = \bigsqcup_{\mathcal{E} \lhd \mathcal{F}} \Delta\left(H^k_{\mathcal{E},-i}\right).$$

Agent $i$ at awareness $\mathcal{F}$ has the set of hierarchies $H_{\mathcal{F},i} = \times_{k \geq 0} H^k_{\mathcal{F},i}$. For each $k$ there is a natural projection from $H_{\mathcal{F},i}$ to $H^k_{\mathcal{F},i}$. The set of hierarchies of all players at all awareness levels is $H = \left(S_{\mathcal{F}}, (H_{\mathcal{F},i})^{i \in I}\right)_{\mathcal{F} \lhd \mathcal{G}}$, with natural projections to the set of order-$k$ hierarchies for all $k$. The universal type space $\Omega$ is mapped to a strict subset of $H$. The construction of the mapping ensures that all elements in the image of $\Omega$ are coherent, i.e. beliefs of different orders do not contradict each other.

The mapping $h = (h_{\mathcal{F},i})^{i \in I_0}_{\mathcal{F} \lhd \mathcal{G}} : \Omega \to H$ from types to hierarchies is also constructed inductively. To define $h_{\mathcal{F},i}$, first $h^k_{\mathcal{F},i} : \Omega \to H^k_{\mathcal{F},i}$ is defined for every $k$. Let $h^k_{\mathcal{F},0} = g_{\mathcal{F},0}$ for every $k$. For $i \in I$, $h^0_{\mathcal{F},i}$ is uniquely defined as the constant function mapping to the singleton $H^0_{\mathcal{F},i}$. Denote $(h^k_{\mathcal{E},i})^{j \in I_0 \setminus i}$ by $h^k_{\mathcal{E},-i}$ and inductively define for each $\mathcal{E} \lhd \mathcal{F} \lhd \mathcal{G}$, $t \in \Omega$ and measurable $F \subseteq H^k_{\mathcal{E},-i}$ the function $h^{k+1}_{\mathcal{F},i}$ as $h^{k+1}_{\mathcal{F},i}(t)(F) = g_i(t)((h^k_{\mathcal{E},-i})^{-1}(F))$, i.e. $h^{k+1} = V(h^k) \circ g$, where $V(h^k)$ is defined in 2. For each $\mathcal{F} \lhd \mathcal{G}$ and $i \in I$, define $h_{\mathcal{F},i} = (h^k_{\mathcal{F},i})_{k \geq 0}$ and $h_{\mathcal{F},0} = g_{\mathcal{F},0}$. This completes the construction of $h$.

The definition of $h$ did not depend on $\Omega$ being universal. The same construction can be used for any type space with unawareness, either directly or by first mapping the type space to the universal space. One way to define the universal space is as those elements of $H$ to which some type in some type space with unawareness is mapped by the $h$ defined on that type space. This is one of the two constructive proofs of the existence of the universal space in Viglizzo (2005).

## 3. Events, belief and awareness

In this section, the form and properties of events are described, where events are those subsets of $\Omega$ that the agents can reason about. First, set operators are defined on general measurable subsets of $\Omega$. Then the set operators are used to inductively define events from measurable sets of states of nature. The model follows Heifetz, Meier, and Schipper (2006) in building unawareness into the structure of the space.

The next subsection defines set operations, belief and awareness in the universal type space with unawareness and constructs events. Belief and awareness are characterized in more detail in subsection 3.2.

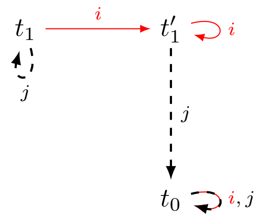



FIGURE 3. The type space with unawareness from the introduction.

3.1. **Conjunction, negation and epistemic operators.** The universal type space with unawareness $\Omega = (\Omega_{\mathcal{F},i})_{i\in I_0}^{\mathcal{F}\triangleleft\mathcal{G}}$ can be divided into layers $\Omega_{\mathcal{F}} = \times_{i\in I_0}\Omega_{\mathcal{F},i}$, one for each awareness level $\mathcal{F} \triangleleft \mathcal{G}$. In Fig. 3, there are two awareness levels: types with subscript 1 are at the higher awareness level, the type $t_0$ is at the lower level.

Associate with each $\Omega_{\mathcal{F}}$ an object $\emptyset_{\mathcal{F}}$, which is the empty set labelled with the awareness level $\mathcal{F}$. It acts as the empty set of the layer $\Omega_{\mathcal{F}}$.

Measurable subsets of $\Omega$ are the disjoint unions of measurable sets, at most one set per layer. Their form is $E = \bigsqcup_{\mathcal{F}\in\zeta} E_{\mathcal{F}}$ for some nonempty $\zeta = \{\mathcal{F}^n : n \in N' \subseteq N\}$, where for each $\mathcal{F} \in \zeta$, the measurable $E_{\mathcal{F}} \subseteq \Omega_{\mathcal{F}}$ is called the $\mathcal{F}$-section of $E$. The operators for conjunction, negation, awareness and belief are defined on the measurable subsets.

The conjunction of sets $E = \bigsqcup_{\mathcal{F}\in\zeta} E_{\mathcal{F}}$ and $F = \bigsqcup_{\mathcal{E}\in\chi} F_{\mathcal{E}}$ is defined as $E \cap F = \bigsqcup_{\mathcal{E}\in\zeta\cap\chi}(E_{\mathcal{E}} \cap F_{\mathcal{E}})$, where it may be the case that $E_{\mathcal{E}} \cap F_{\mathcal{E}} = \emptyset_{\mathcal{E}}$ for some layers $\mathcal{E}$. The negation of $E$ is $E^{c'} = \bigsqcup_{\mathcal{F}\in\zeta}(\Omega_{\mathcal{F}} \setminus E_{\mathcal{F}})$. It satisfies $(E^{c'})^{c'} = E$, but is not equal to $E^c$, the complement of $E$ in $\Omega$. The nonstandard negation is what allows the present model to bypass the Dekel, Lipman, and Rustichini (1998) impossibility result stating that standard state spaces preclude unawareness. This is discussed in more detail in section 4.

Disjunction of sets is defined from conjunction and negation via De Morgan's laws: $E \cup' F = (E^{c'} \cap F^{c'})^{c'}$. Equality of sets $E$ and $F$ requires $\zeta = \chi$ and $E_{\mathcal{E}} = F_{\mathcal{E}}$ for each $\mathcal{E} \in \zeta$. Set $E$ implies $F$ (is a subset of $F$), written as $E \subseteq' F$ if $\zeta = \chi$ and $E_{\mathcal{E}} \subseteq F_{\mathcal{E}}$ for each $\mathcal{E} \in \zeta$. Equality and implication respect the usual propositional tautologies: $E = F$ is the same as $E \cap F \cup' (E^{c'} \cap F^{c'})$, and $E \subseteq' F$ is the same as $E^{c'} \cup' F$.[2]

Operations on sets are illustrated using the type space corresponding to Feinberg's game in Fig. 3. Abusing notation, we denote the singleton sets $\{t\}$ by $t$. The negation of $t_1$ is $t_1^{c'} = t'_1$, not the complement $t_1^c = \{t'_1, t_0\}$. The disjunction of $t_1$ and $\{t'_1, t_0\}$ is $t_1 \cup' \{t'_1, t_0\} = \{t_1, t'_1\}$, not the set-theoretic union $\{t_1, t'_1, t_0\}$.

The set $B_r^i E$ in which agent $i$ puts probability at least $r$ on $E$ consists of measurable sets from the layers $\Omega_{\mathcal{D}}$ that contain more awareness than some layer in which $E$ has a section. For each $\mathcal{D}$ that is a refinement of some $\mathcal{F} \in \zeta$, the $\mathcal{D}$-section of $B_r^i E$ is made up of those types in $\Omega_{\mathcal{D}}$ that put probability at least $r$ on some section of $E$. Formally,

$$B_r^i E = \bigsqcup_{\mathcal{D}\triangleright\mathcal{F}\in\zeta} \{t \in \Omega_{\mathcal{D}} : \exists \mathcal{F} \in \zeta,\ g_i(t) \in \Delta(\Omega_{\mathcal{F},-i}),\ g_i(t)(E_{\mathcal{F}}) \geq r\}. \tag{4}$$

In Fig. 3, a belief set for $i$ is $B_1^i t'_1 = \{t_1, t'_1\}$ and a (non)belief set for $j$ is $B_{1/3}^j t'_1 = \emptyset_1$.

The set in which agent $i$ is aware of $E$ is defined as

$$A^i E = \bigsqcup_{\mathcal{D}\triangleright\mathcal{F}\in\zeta} \{t \in \Omega_{\mathcal{D}} : \exists \mathcal{F} \in \zeta,\ g_i(t) \in \Delta(\Omega_{\mathcal{F},-i})\}. \tag{5}$$

In Fig. 3, agent $i$'s awareness sets are for example $A^i t_1 = A^i t'_1 = \{t_1, t'_1\}$ and one of agent $j$'s unawareness sets is $(A^j t_1)^{c'} = t'_1$.

Events are measurable subsets of $\Omega$ that have the form $E = \bigsqcup_{\mathcal{F}\triangleright\mathcal{E}} E_{\mathcal{F}}$, though not all sets of this form need be events. The definition of events is inductive. The base case is to take for any measurable set of states of nature $E_S \in \mathcal{G}_S$ the coarsest $\mathcal{E}_S$ in which $E_S$ is measurable and let $E = \bigsqcup_{\mathcal{F}\triangleright\mathcal{E}} E_{\mathcal{F}}$, where $E_{\mathcal{F}} = E_S \times \times_{i\in I}\Omega_{\mathcal{F},i}$. The inductive step is to let for any events $E$ and $F$, any $i \in I$ and $r \in [0,1]$ the sets $E^{c'}$, $E \cap F$, $B_r^i E$ and $A^i E$ also be events.

[2] An alternative definition of implication is: $E \subseteq F$ iff $\zeta \subseteq \chi$ and $E_{\mathcal{E}} \subseteq F_{\mathcal{E}}$ for each $\mathcal{E} \in \zeta$. This follows the intuition of the subset relation. $E \subseteq F$ is not equal to $E^{c'} \cup' F$, but $E \subseteq F \cap F \subseteq E$ is the same as $E = F$. Since $\subseteq$ is weaker than $\subseteq'$, all the results for $\subseteq'$ below continue to hold for $\subseteq$.

The definition of events is reminiscent of the construction of a language (a syntax) in modal logic. In logic, each formula is assigned a set of states where it is true after the full language has been defined, while here the semantics and syntax are created together in the same induction.

3.2. **Properties of belief and awareness.** This subsection characterizes belief and awareness in type spaces with unawareness in more detail, without assuming that the sets the operators are applied to are events. Events are a subclass of the sets considered, so all results continue to hold for events.

*Remark* 1. Three observations following directly from the definitions of belief and awareness are

(1) $A^iE = B^i_0E = B^i_r(E \cup' E^{c'}) = A^i(E^{c'})$,
(2) $B^i_rE \subseteq' B^i_qE$ for all $r \geq q$, in particular, $B^i_rE \subseteq' A^iE \quad \forall r$,
(3) $B^i_rE \subseteq' (B^i_q(E^{c'}))^{c'}$ for $r + q > 1$.

Awareness is symmetric with respect to negation in this model: $A^iE = A^i(E^{c'})$, therefore unawareness is also symmetric. Awareness is belief with probability at least zero, which is distinct from belief with probability exactly zero (i.e. the negation having probability one). Belief with probability exactly zero is not symmetric: $B^i_0E \cap B^i_1(E^{c'}) \neq B^i_0(E^{c'}) \cap B^i_1(E)$.

In the universal type space, the sets $A^iE$ and $B^i_rE$ contain a measurable set from each $\Omega_{\mathcal{D}}$ for which $\mathcal{D} \triangleright \mathcal{F}$ for some $\mathcal{F} \in \zeta$, because every layer of the universal space contains a belief over each layer with less awareness. If the $\mathcal{F}$-section of $E$ is nonempty, then the $\mathcal{D}$-sections of $A^iE$ and $B^i_rE$ are also nonempty, because each layer of the universal space contains all beliefs over layers with weakly less awareness, i.e. layer $\Omega_{\mathcal{F}}$ contains for any nonempty set $E_{\mathcal{E}}$ with $\mathcal{E} \triangleleft \mathcal{F}$ a type putting probability one on $E_{\mathcal{E}}$. Probability one implies probability at least $r$ for any $r \in [0,1]$ by Remark 1.

Unawareness is nontrivial in this paper, because if $E = \bigsqcup_{\mathcal{F} \in \zeta} E_{\mathcal{F}}$ and $\mathcal{E} \notin \zeta$, then there exists a type $t$ with $g_i(t) \in \Delta(\Omega_{\mathcal{E},-i})$ . That type is unaware of set $E$, so $(A^iE)^{c'}$ is nonempty. Since $\zeta$ must be nonempty, $A^iE$ contains a measurable set from at least one layer of $\Omega$. If $E$ contains at least one nonempty set, then at least one measurable set in $A^iE$ is nonempty. Unlike in Dekel, Lipman, and Rustichini (1998), unawareness is generally neither the whole space nor the empty set, and the same holds for belief.

Next it is shown that belief with probability one and awareness satisfy the conjunction property—the conjunction of sets is certain iff both sets are certain and the agent is aware of the conjunction of sets iff he is aware of both. The proof is purely from the definitions.

**Lemma 2.** *$B^i_1E \cap B^i_1F = B^i_1(E \cap F)$ and $A^iE \cap A^iF = A^i(E \cap F)$*

*Proof.* Take $E = \bigsqcup_{\mathcal{F} \in \zeta} E_{\mathcal{F}}$, $F = \bigsqcup_{\mathcal{E} \in \chi} F_{\mathcal{E}}$. Then

$$\begin{aligned}
B^i_1E \cap B^i_1F &= \bigsqcup_{\mathcal{D} \triangleright \mathcal{F} \in \zeta \cap \chi} \left(\{t \in \Omega_{\mathcal{D}} : \exists \mathcal{F} \in \zeta \cap \chi,\ g_i(t)(E_{\mathcal{F}}) \geq 1\} \cap \{t \in \Omega_{\mathcal{D}} : \exists \mathcal{F} \in \zeta \cap \chi,\ g_i(t)(F_{\mathcal{F}}) \geq 1\}\right) \\
&= \bigsqcup_{\mathcal{D} \triangleright \mathcal{F} \in \zeta \cap \chi} \{t \in \Omega_{\mathcal{D}} : \exists \mathcal{F} \in \zeta \cap \chi,\ g_i(t)(E_{\mathcal{F}} \cap F_{\mathcal{F}}) \geq 1\} = B^i_1(E \cap F) \\
A^iE \cap A^iF &= \bigsqcup_{\mathcal{D} \triangleright \mathcal{F} \in \zeta \cap \chi} \left(\{t \in \Omega_{\mathcal{D}} : \exists \mathcal{F} \in \zeta \cap \chi,\ g_i(t) \in \Delta(\Omega_{\mathcal{F},-i})\} \cap \{t \in \Omega_{\mathcal{D}} : \exists \mathcal{F} \in \zeta \cap \chi,\ g_i(t) \in \Delta(\Omega_{\mathcal{F},-i})\}\right) \\
&= A^i(E \cap F)
\end{aligned}$$

□

Belief with probability $r \in (0,1)$ does not satisfy conjunction. An easy counterexample is the uniform distribution on $\{a,b,c\}$ which puts probability at least $\frac{2}{3}$ on both $\{a,b\}$ and $\{b,c\}$, but not on $\{b\}$.

Probabilistic belief does satisfy two conjunction-like properties similar to those given in Samet (2000) and Zhou (2009). These are

$$B^i_r(E \cap F) \cap B^i_q(E \cap F^{c'}) \subseteq' B^i_{r+q}(E \cap (F \cup' F^{c'})) = B^i_{r+q}E \cap A^iF, \quad r + q \leq 1,$$

$$A^iE \cap A^iF \cap \left(B^i_r(E \cap F)\right)^{c'} \cap \left(B^i_q(E \cap F^{c'})\right)^{c'} \subseteq' \left(B^i_{r+q}E\right)^{c'}, \quad r + q \leq 1.$$

Awareness must be added to the original conditions to make all sets consist of sections from the same layers of $\Omega$.

Subsequently only sets of measurable sets of the form $E = \bigsqcup_{\mathcal{F} \triangleright \mathcal{E}} E_{\mathcal{F}}$ are considered, where $\mathcal{E}$ can be any coarsening of $\mathcal{G}$. For sets of the form $E = \bigsqcup_{\mathcal{F} \triangleright \mathcal{E}} E_{\mathcal{F}}$, define the base space as the coarsest space containing some section of $E$, i.e. $\Omega(E) = \Omega_{\mathcal{E}}$ for the $E$ in this paragraph. Applying the belief operator to a set of this form results in another set of this form with the same base space, as shown in Lemma 3. The same holds for the awareness operator by Remark 1, and for negation and conjunction by definition.

**Lemma 3.** *In the universal space with unawareness, if* $E = \bigsqcup_{\mathcal{F} \triangleright \mathcal{E}} E_{\mathcal{F}}$*, then* $B^i_r E = \bigsqcup_{\mathcal{F} \triangleright \mathcal{E}} (B^i_r E)_{\mathcal{F}}$ *for the same* $\mathcal{E}$.

*Proof.* By the definition of $B^i_r E$ in Eq. (4), for each $E_{\mathcal{E}} \in E$ and $\mathcal{F} \triangleright \mathcal{E}$, $B^i_r E$ contains a measurable set from each layer $\Omega_{\mathcal{F}}$. In particular, $B^i_r E$ contains a subset of each layer with more awareness than the base space of $E$. Again by (4), $B^i_r E$ does not contain a set from any layer $\Omega_{\mathcal{D}}$ that does not have weakly more awareness than some section $E_{\mathcal{F}'}$ of $E$, i.e. for which $\mathcal{D} \triangleright \mathcal{F}'$ fails. So $B^i_r E$ does not contain a subset of any layer that does not have weakly more awareness than the base space of $E$. □

*Remark* 2. If $\Omega(E) = \Omega(F) = \Omega_{\mathcal{E}}$ and $E_{\mathcal{F}} \subseteq F_{\mathcal{F}}\ \forall \mathcal{F} \triangleright \mathcal{E}$, then $(B^i_r E)_{\mathcal{F}} \subseteq (B^i_r F)_{\mathcal{F}}\ \forall \mathcal{F} \triangleright \mathcal{E}$, which is equivalent to $B^i_r E \subseteq' B^i_r F$. By Remark 1 and Lemma 3, it follows that $B^i_q B^i_r E \subseteq' B^i_q B^i_p E$ for all $r \geq p$.

The next proposition captures the positive introspection property—if an agent believes something, then he is certain that he believes it. This property relies directly on the specific extension of $g_i$ to the $i$-th dimension presented in subsection 2.2. The extension builds certainty of own belief into the definition of the type function.

**Proposition 4.** $B^i_r E = B^i_1 B^i_r E \quad \forall i\ \forall r\ \forall E$

*Proof.* For the set $B^i_r E$, only the $i$-th dimension of a type is important, in the following sense. A type $t$ can expressed as $(t_i, t_{-i})$. By the definition of the belief operator in (4), the set $B^i_r E$ consists of the types $(t_i, t_{-i})$ where $g_i(t_i)(E) \geq r$ (for this, $E$ must be defined at the awareness level $\mathcal{E}$ of $t_i$). The component $t_{-i}$ of the type is unrestricted, except by the requirement that $(t_i, t_{-i})$ belong to the appropriate layer of the space. So if $(t_i, t_{-i}) \in B^i_r E$, then also $(t_i, t'_{-i}) \in B^i_r E$ for any $t'_{-i}$ from the same layer of the space.

The probability $g_i(t)(B^i_r E)$ that $i$ puts on $B^i_r E$ depends only on the marginal $\delta_{t_{\mathcal{E},i}}$: if $(t_{\mathcal{E},i}, t_{\mathcal{E},-i})$ is contained in $B^i_r E$ for some $t_{\mathcal{E},-i}$, then the probability is one, otherwise zero. This is formally expressed in the following string of equivalences, which establishes the result.

$$g_i(t_i)(B^i_r E) \geq 1 \Leftrightarrow (t_{\mathcal{E},i}, t_{\mathcal{E},-i}) \in B^i_r E \Leftrightarrow g_i(t_{\mathcal{E},i})(E) \geq r \Leftrightarrow g_i(t_i)(E) \geq r$$

The last equivalence comes from the fact that $g_i(t_i) = g_i(t_{\mathcal{E},i})$. □

From Remark 1 and Proposition 4, $A^i E = B^i_1 A^i E$ and $A^i E \subseteq' A^i A^i E$ follow immediately. The latter implication is strengthened to equality in the next proposition, which demonstrates that the universal type space with unawareness satisfies the AU introspection property of Dekel, Lipman, and Rustichini (1998).

**Proposition 5.** $(A^i E)^{c'} = (A^i (A^i E)^{c'})^{c'}$ *and* $A^i E = A^i A^i E = A^i B^i_r E$.

*Proof.* It is true that $t \in (A^i (A^i E)^{c'})^{c'}$ iff it is not true that $g_i(t) \in \Delta(\Omega_{\mathcal{E},-i})$ s.t. $\mathcal{E} \triangleright \mathcal{F}$, where $\Omega((A^i E)^{c'}) = \Omega_{\mathcal{F}}$. By Lemma 3 and Remark 1, $\Omega((A^i E)^{c'}) = \Omega(A^i E) = \Omega(E)$. Therefore $t \in (A^i (A^i E)^{c'})^{c'}$ iff $g_i(t) \notin \Delta(\Omega_{\mathcal{E},-i})$ s.t. $\mathcal{E} \triangleright \mathcal{F}$, where $\Omega(E) = \Omega_{\mathcal{F}}$. This is equivalent to $t \in (A^i E)^{c'}$.

For the second part, $t \in (A^i E)_{\mathcal{F}}$ iff $g_i(t) \in \Delta(\Omega_{\mathcal{F}',-i})$ for some $\mathcal{F}' \triangleright \mathcal{E}$, where $\Omega(E) = \Omega_{\mathcal{E}}$. Note that $\Omega(E) = \Omega(A^i E) = \Omega(B^i_r E)$ by Lemma 3, therefore $g_i(t) \in \Delta(\Omega_{\mathcal{F}',-i})$ iff $t \in (A^i A^i E)_{\mathcal{F}}$ iff $t \in (A^i B^i_r E)_{\mathcal{F}}$. □

Lemma 6 proves the KU introspection property, which similarly to AU introspection comes from Dekel, Lipman, and Rustichini (1998) and is used in section 4.

**Lemma 6.** $B^i_1 (A^i E)^{c'} = \emptyset$.

*Proof.* By Remark 1 and by Proposition 5, $B^i_1 (A^i E)^{c'} \subseteq' A^i (A^i E)^{c'} = A^i E$. Together, $B^i_1 (A^i E)^{c'} \subseteq' (A^i E)^{c'}$ and $B^i_1 (A^i E)^{c'} \subseteq' A^i E$ give $B^i_1 (A^i E)^{c'} = \bigsqcup_{\mathcal{F} \triangleright \mathcal{E}} \emptyset_{\mathcal{F}}$, where $\Omega(E) = \Omega_{\mathcal{E}}$. □

The negative introspection property $(B^i_r E)^{c'} \subseteq' B^i_q (B^i_r E)^{c'}$ fails for any $q \in [0, 1]$, because $(B^i_r E)^{c'}$ contains types that are unaware of $E$ and therefore unaware of $(B^i_r E)^{c'}$. However, it can be shown that conditional on being aware, negative introspection holds.

**Proposition 7.** $A^i E \cap (B^i_r E)^{c'} = B^i_1 (B^i_r E)^{c'}$

*Proof.* As in Proposition 4, only the $t_i$ dimension of types matters. The result is established by the following equivalences.

$$g_i(t_i)((B^i_r E)^{c'}) \geq 1 \Leftrightarrow (t_{\mathcal{E},i}, t_{\mathcal{E},-i}) \in (B^i_r E)^{c'} \text{ and } (t_{\mathcal{E},i}, t_{\mathcal{E},-i}) \in A^i E \Leftrightarrow 0 \leq g_i(t_{\mathcal{E},i})(E) < r \Leftrightarrow 0 \leq g_i(t_i)(E) < r.$$

□

The next section demonstrates that the desirable properties of unawareness and certainty are satisfied in the universal type space with unawareness, while unawareness is nontrivial. Therefore the model of this paper escapes the impossibility result of Dekel, Lipman, and Rustichini (1998).

## 4. Evading the impossibility result of Dekel, Lipman, and Rustichini (1998)

Every unawareness model must deal with the limitations imposed by the impossibility result that standard state spaces preclude unawareness. In this section it is shown that the universal type space with unawareness is not a standard state space model according to the definition of Dekel, Lipman, and Rustichini and permits nontrivial unawareness. First, an overview of the impossibility result is in order.

4.1. **Overview of Dekel, Lipman, and Rustichini (1998).** There is a knowledge operator $K$ and an unawareness operator $U$ on a state space $\Omega$. The model has a single agent. The axioms described below are imposed on a state space model and the following theorem is proved.

A1 (necessitation) $K\Omega = \Omega$

A2 (monotonicity) $E \subseteq F$ implies $KE \subseteq KF$

A4 (plausibility): $UE \subseteq (KE)^c \cap (K(KE)^c)^c$

A5 (KU introspection): $KUE = \emptyset$

A6 (AU introspection): $UE \subseteq UUE$. Equivalently $AUE \subseteq AE$, where $AE = (UE)^c$ is the awareness operator.

**Theorem 8** (Theorem 1 of Dekel, Lipman, and Rustichini)**.** *A4-A6, A1 imply $UE = \emptyset$ (never unaware).*
*A4-A6, A2 imply $UE \subseteq (KF)^c$ (unawareness means no knowledge of anything).*

*Proof.* $UE \subseteq UUE \subseteq (K(KUE)^c)^c = (K\Omega)^c$. A1 is $K\Omega = \Omega$, A2 implies $(K\Omega)^c \subseteq (KF)^c$. □

The preceding theorem applies to set-based models, but using a propositional model does not necessarily permit nontrivial unawareness, as the next theorem shows. The notation for a propositional model uses formulas $\phi$, $\psi$, and redefines the knowledge and unawareness operators to map formulas to formulas. There are logic operators negation $\neg$ and disjunction $\vee$ (corresponding to complement and union) also taking formulas to formulas. The function $\|\cdot\| : \mathcal{G} \to 2^\Omega$ gives for each formula $\phi$ the set of states in which $\phi$ is true. The additional axioms on propositional models are as follows.

A7 (real states): $\|\phi\| = \Omega \setminus \|\neg\phi\|$, $\quad \|\phi \vee \neg\phi\| = \Omega$

A8 (event sufficiency): $\|\phi\| = \|\psi\|$ implies $\|K\phi\| = \|K\psi\|$ and $\|U\phi\| = \|U\psi\|$

A9 (weak necessitation): $\|\neg U\phi\| \subseteq \|K(\phi \vee \neg\phi)\|$, equivalently $(UE)^c \subseteq K\Omega$.

Under A7 and A8, a propositional model can be expressed as a standard state space model by setting $KE = \|K\phi\|$ and $UE = \|U\phi\|$ for any $\phi$ with $\|\phi\| = E$, so Theorem 8 applies. The following theorem uses slightly weaker axioms than Theorem 8 and proves a similar result.

**Theorem 9** (Theorem 2 of Dekel, Lipman, and Rustichini)**.** *A4-A9 imply $UE = UF \subseteq (KG)^c$.*

*Proof.* By Theorem 8, $UE \subseteq (K\Omega)^c$. A9 gives $(K\Omega)^c \subseteq UF$, so $UE \subseteq UF$. A4 gives $UG \subseteq (KG)^c$. □

The next subsection shows that the universal type space with unawareness satisfies the properties of knowledge and unawareness considered natural in Dekel, Lipman, and Rustichini (1998), while describing nontrivial unawareness.

4.2. **Connection to type spaces with unawareness.** To discuss the theorems of Dekel, Lipman, and Rustichini (1998) in the model of this paper, pick any agent $i$ and take knowledge to mean $i$'s belief with probability one, $KE = B_1^i E$. Complementation $^c$ in the preceding subsection will correspond to negation $^{c'}$ in the universal space. Unawareness is the negation of awareness for that agent, $UE = (A^i E)^{c'}$. There are two possible interpretations of Dekel, Lipman, and Rustichini's state space $\Omega$ in a type space with unawareness—the whole type space $\Omega$ or a collection $\{\Omega_{\mathcal{F}} : \mathcal{F} \triangleright \mathcal{E}\}$ for some $\mathcal{E}$. Necessitation holds under the first interpretation, but not necessarily under the second, since for $\mathcal{D} \triangleleft \mathcal{E}$, types in $\Omega_{\mathcal{D}}$ do not form beliefs about any $\Omega_{\mathcal{F}}$ with $\mathcal{F} \triangleright \mathcal{E}$.

Monotonicity holds in type spaces with unawareness, because a given type needs weakly more awareness to reason about $E \subseteq' F$ than about $F$ and puts weakly less probability on $E$. If $E$ has probability one for a type, then the type is aware of $F$ and puts probability one on $F$.

The first part of plausibility, $(A^i E)^{c'} \subseteq' (B_1^i E)^{c'}$, follows from Equations (4) and (5) defining belief and awareness. The second part, $(A^i E)^{c'} \subseteq' (B_1^i (B_1^i E)^{c'})^{c'}$, is implied by Proposition 7. The KU introspection property is exactly Lemma 6 in subsection 3. AU introspection follows from Proposition 5, which actually gives equality instead of a subset relation.

The crucial property of the universal type space with unawareness is the failure of the real states axiom of Dekel, Lipman, and Rustichini (1998)—the negation of an event is generally not equal to the complement of the event, and tautologies $E \cap E^{c'}$ that use the nonstandard negation do not equal the whole $\Omega$. Since the belief and awareness operators in the universal space are set-based, they satisfy the event sufficiency axiom.

Weak necessitation is implied by necessitation, so it holds under the interpretation that equates Dekel, Lipman, and Rustichini's state space with the whole type space. Under the interpretation that makes the state space correspond to $\{\Omega_{\mathcal{F}} : \mathcal{F} \triangleright \mathcal{E}\}$ for some $\mathcal{E}$, weak necessitation may hold, even though necessitation fails. It depends on the particular $\mathcal{E}$ chosen. The natural choice, based on the $\|\neg U\phi\| \subseteq \|K(\phi \vee \neg\phi)\|$ version of weak necessitation, is to set $\mathcal{E}$ equal to the base space of the event considered, i.e. to write the second version of weak necessitation as $(UE)^{c'} \subseteq' K(E \cup' E^{c'})$. This makes weak necessitation hold.

The properties of knowledge and unawareness that Dekel, Lipman, and Rustichini considered desirable (plausibility, KU introspection and AU introspection) are satisfied in the model of this paper. Of the additional properties they considered, monotonicity and event sufficiency hold unconditionally, while necessitation and weak necessitation hold under some interpretations, but not others. The impossibility theorems of Dekel, Lipman, and Rustichini (1998) fail due to the failure of the real states assumption—as was argued in section 3, unawareness and belief are nontrivial.

## 5. Other kinds of unawareness

Unawareness in the universal type space does not have to be propositionally generated, as in the preceding sections. The same proof of existence and characterization of events, and similar properties of belief and awareness hold if the awareness levels are subsets $J$ of the finite set $I$ of agents. In that case, an agent could reason only about the agents he is aware of, their reasoning about the agents he is aware of etc.

Bounded reasoning in the style of Kets (2010) can be expressed as unawareness of higher-order reasoning by taking the awareness levels to be $1 \leq k \leq \infty$. An agent with awareness $k$ can reason about order $k-1$ and lower order beliefs of other agents, their reasoning about order $k-2$ and lower orders etc. The special case of infinite depth of reasoning is taken care of by assuming $\infty - 1 = \infty$.

The proof of existence of the universal space in Viglizzo (2005) and Moss and Viglizzo (2006) only deals with a finite number of agents, and mathematically the fixed set of awareness levels in this paper behaves like the set of agents. With unawareness of higher-order reasoning, the set of awareness levels is countably infinite. However, since all the proofs of Moss and Viglizzo (2006) use measurable spaces and the assumption about a finite number of agents is only used to make showing measurability easier, the proofs work with a countable number of awareness levels as well.[3] The events and the belief and awareness operators have a similar structure to section 3 if the awareness levels are orders of reasoning.

If the existence result holds for several kinds of unawareness, it also holds for their combination, with only notational changes in the proof. Type spaces with three kinds of unawareness consist of measurable spaces $M_{\mathcal{F},J,k,i}$ one for each agent $i$ and awareness level $(\mathcal{F}, J, k)$. The triple consists of a $\sigma$-algebra $\mathcal{F} \triangleleft \mathcal{G}$ on the states of nature, a set of agents $J \subseteq I$ and an order of reasoning $k$. Each type in $M_{\mathcal{F},J,k,i}$ can reason only about the agents in $J$ and propositions in $\mathcal{F}$, and the reasoning can extend only up to order $k$.

The definitions of type spaces with different kinds of unawareness are given next, followed by the proof of the existence of the universal type space for all of them.

**Definition 4.** A type space with unawareness of agents is $(M, g) = \left(M_0, (M_{J,i})_{J\subseteq I}^{i\in J},\ g_0, (g_{J,i})_{J\subseteq I}^{i\in J}\right)$ such that

(i) $M_{J,i}$ is a measurable space for each $J$ and $i$, and $M_0 = S_{\mathcal{G}}$,
(ii) for each $J$ and $i$, the type function $g_{J,i} : M_{J,i} \to \bigsqcup_{i\in J'\subseteq J} \Delta\left(M_0 \times \times_{j\in J'\setminus\{i\}} M_{J',j}\right)$ is measurable,
(iii) the state of nature function $g_0 : M_0 \to S_{\mathcal{G}}$ is the identity.

The dimension of the type morphism must be modified to capture the different number of components of a type space with unawareness of agents, but the definition remains basically the same. The definition of the universal type space is the same for all kinds of unawareness—it is the space that has a unique type morphism from every type space with the same kind of unawareness into it. The existence of the universal type space for all kinds of unawareness in this section is given in Theorem 10. Before stating the theorem, type spaces are defined for the remaining kinds of unawareness considered in this paper.

**Definition 5.** A type space with unawareness of higher-order reasoning is

$$(M, g) = \left(M_0, (M_{k,i})_{1\leq k\leq\infty}^{i\in I},\ g_0, (g_{k,i})_{1\leq k\leq\infty}^{i\in I}\right)$$

[3] This was confirmed by Larry Moss in a private communication.

such that

(i) $M_{k,i}$ is a measurable space for each $k$ and $i$, and $M_0 = S_{\mathcal{G}}$,
(ii) for each $1 \leq k \leq \infty$ and $i$, the type function $g_{k,i} : M_{k,i} \to \bigsqcup_{n \leq k-1} \Delta\left(M_0 \times \times_{j \in I \setminus \{i\}} M_{n,j}\right)$ is measurable, where $M_{0,i}$ is a singleton,
(iii) the state of nature function $g_0 : M_0 \to S_{\mathcal{G}}$ is the identity.

**Definition 6.** A type space with three kinds of unawareness is

$$(M, g) = \left(\left(M_{\mathcal{F},0}, (M_{\mathcal{F},J,k,i})^{i \in J \subseteq I}_{1 \leq k \leq \infty}\right)_{\mathcal{F} \lhd \mathcal{G}}, \left(g_{\mathcal{F},0}, (g_{\mathcal{F},J,k,i})^{i \in J \subseteq I}_{1 \leq k \leq \infty}\right)_{\mathcal{F} \lhd \mathcal{G}}\right)$$

such that

(i) $M_{\mathcal{F},J,k,i}$ is a measurable space for each $\mathcal{F}$, $J$, $k$ and $i$, and $M_{\mathcal{F},0} = S_{\mathcal{F}}$ for each $\mathcal{F}$,
(ii) for each $\mathcal{F}$, $J$, $k \geq 1$ and $i$, the type function

$$g_{\mathcal{F},J,k,i} : M_{\mathcal{F},J,k,i} \to \bigsqcup_{\substack{\mathcal{E} \lhd \mathcal{F} \\ i \in J' \subseteq J \\ n \leq k-1}} \Delta\left(M_{\mathcal{E},0} \times \times_{j \in J' \setminus \{i\}} M_{\mathcal{E},J',n,j}\right)$$

is measurable, with $M_{\mathcal{E},J',0,i}$ a singleton,
(iii) for each $\mathcal{F}$, the state of nature function $g_{\mathcal{F},0} : M_{\mathcal{F},0} \to S_{\mathcal{F}}$ is the identity.

**Theorem 10.** *For fixed sets of states of nature, agents and awareness levels, there exists a universal type space with unawareness of agents $(\Omega^a, g^a)$, a universal type space with unawareness of higher-order reasoning $(\Omega^b, g^b)$ and a universal type space with three kinds of unawareness $(\Omega^c, g^c)$. All universal type spaces are unique up to isomorphism. The type functions in the universal type spaces are isomorphisms.*

The proof in the appendix is very similar to Theorem 1. It also translates the problem to category theory and uses the results of Viglizzo (2005).

In the next section, I consider the case where the set of states of nature is a Polish space. This additional assumption allows a different construction of the universal type space that is more common in the literature (Mertens and Zamir, 1985; Brandenburger and Dekel, 1993; Heifetz, Meier, and Schipper, 2011b).

## 6. The topological case

Previously, the only assumption on the set of states of nature was that they form a measurable space. All probability measures over $S$ were allowed as beliefs, all probability measures over these as beliefs about beliefs, etc. This permitted a construction of the universal type space as all hierarchies that some type in some type space maps to. Before Heifetz and Samet (1998b), the proofs of the existence of the universal type space used topological assumptions and applied the Kolmogorov consistency theorem to show that the set of all consistent hierarchies constituted the universal type space. For this proof, the states of nature need to have a topology and the set of beliefs must be the set Borel probability measures with the topology of weak convergence.

In this subsection, the older method is used to prove the existence of universal type spaces for the kinds of unawareness considered in this paper. First, the definitions of type spaces and type morphisms are modified to incorporate topological assumptions.

Let $S$ be a Polish space (a complete separable metric space) with the Borel $\sigma$-algebra generated by the open sets of the topology induced by the metric. For any Polish space $M$, let $\Delta(M)$ denote the set of all Borel probability measures on $M$, with the $\sigma$-algebra on $\Delta(M)$ generated by

$$\{\beta^q(E) : E \in \mathcal{G}_M,\ q \in [0,1]\},\ \text{where } \beta^q(E) = \{\mu \in \Delta(M) : \mu(E) \geq q\}.$$

The rest of the notation from section 2 has the same meaning as previously. The definitions of type spaces and type morphisms now require all spaces to be Polish and measurable with the Borel $\sigma$-algebra. All functions in the definitions are required to be continuous and measurable. All other aspects of the definitions remain the same as in sections 2 and 5. The following theorem establishes that for the kinds of unawareness examined in this paper, the set of all coherent hierarchies forms the universal type space.

**Theorem 11.** *For a Polish space of states of nature, a countable set of agents and awareness levels, there exist universal type spaces with propositional unawareness, unawareness of agents, unawareness of higher-order reasoning and with three kinds of unawareness. All universal type spaces are unique up to isomorphism. The type functions in the universal type spaces are isomorphisms.*

For the case of propositional unawareness, the proof is in Heifetz, Meier, and Schipper (2011b). The proofs for the other kinds of unawareness use the general theorem of Schubert (2008) and are presented in the appendix.

The mechanism by which the universal type space of the present paper achieves nontrivial unawareness is the same as in Heifetz, Meier, and Schipper (2006, 2011b), so it is useful to compare the approaches. The next subsection gives an overview of Heifetz, Meier, and Schipper's model and compares it to the universal space with unawareness.

6.1. **Comparison to Heifetz, Meier, and Schipper (2011b).** The universal type space with propositional unawareness in Heifetz, Meier, and Schipper (2011b) is based on topological spaces and uses the Kolmogorov extension theorem. The construction starts from a complete lattice of spaces of states of nature, which corresponds to the present paper's lattice of $\sigma$-algebras on the states of nature. Both will be denoted $(S_{\mathcal{F}})_{\mathcal{F}\lhd\mathcal{G}}$. The notation differs from Heifetz, Meier, and Schipper in many respects, in order to use the same symbols for corresponding objects and different symbols for different objects.

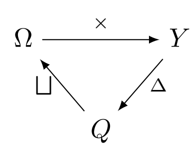
The hierarchical construction of Heifetz, Meier, and Schipper sets $Y^0_{\mathcal{F},i} = S_{\mathcal{F}}$ and defines the first order beliefs of player $i$ as $Q^1_{\mathcal{F},i} = \Delta(Y^0_{\mathcal{F},i})$, the space of regular Borel probability measures on $Y^0_{\mathcal{F},i}$ endowed with the topology of weak convergence. Inductively, the domain of $(k+1)$-order beliefs is $Y^k_{\mathcal{F},i} = S_{\mathcal{F}} \times \times_{j\neq i} \left(\bigsqcup_{\mathcal{E}\lhd\mathcal{F}} Q^k_{\mathcal{E},j}\right)$ and the set of consistent $(k+1)$-order beliefs is

$$Q^{k+1}_{\mathcal{F},i} = \left\{(\mu^1,\dots,\mu^{k+1}) \in Q^k_{\mathcal{F},i} \times \Delta\left(Y^k_{\mathcal{F},i}\right) : \mathrm{mrg}_{Y^{k-1}_{\mathcal{F},i}} \mu^{k+1} = \mu^k\right\}.$$

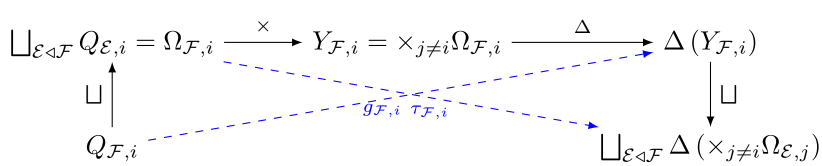
In the limit, define $Q_{\mathcal{F},i} = \left\{(\mu^1,\mu^2,\dots) : (\mu^1,\dots,\mu^k) \in Q^k_{\mathcal{F},i}\ \forall k\right\}$ and $Y_{\mathcal{F},i} = S_{\mathcal{F}} \times \times_{j\neq i} \left(\bigsqcup_{\mathcal{E}\lhd\mathcal{F}} Q_{\mathcal{E},j}\right)$. The belief function of agent $i$ at awareness $\mathcal{F}$ is $\tau_{\mathcal{F},i} : Q_{\mathcal{F},i} \to \Delta\left(Y_{\mathcal{F},i}\right)$. The full type space is $(Y_{\mathcal{F}})_{\mathcal{F}\lhd\mathcal{G}}$, where $Y_{\mathcal{F}} = S_{\mathcal{F}} \times \times_{i\in I} \left(\bigsqcup_{\mathcal{E}\lhd\mathcal{F}} Q_{\mathcal{E},i}\right)$. The belief function is redefined to $\tau_i : Y_{\mathcal{F}} \to \bigsqcup_{\mathcal{E}\lhd\mathcal{F}} \Delta\left(Y_{\mathcal{E}}\right)$, but it still depends only on the $i$-th coordinate of $Y_{\mathcal{F}}$. Beliefs are extended from $Y_{\mathcal{F},i}$ to $Y_{\mathcal{F}}$ by imposing certainty of own belief (taking the Cartesian product with the Dirac $\delta$-function on own type), similarly to subsection 2.2 of the present paper.

Starting from the same states of nature, agents and awareness levels, the full type space $(Y_{\mathcal{F}})_{\mathcal{F}\lhd\mathcal{G}}$ of Heifetz, Meier, and Schipper equals $(\Omega_{\mathcal{F}})_{\mathcal{F}\lhd\mathcal{G}}$ in the topological part of the present paper. The type function $\tau_i$ maps each element of $Y_{\mathcal{F}}$ to $\Delta\left(Y_{\mathcal{E}}\right)$ for some $\mathcal{E}\lhd\mathcal{F}$, just like $g_i$ maps $\Omega_{\mathcal{F}}$ to $\bigsqcup_{\mathcal{E}\lhd\mathcal{F}} \Delta\left(\Omega_{\mathcal{E}}\right)$. Therefore the category of type spaces with unawareness is the same the present paper as in Heifetz, Meier, and Schipper (2011b). This is not surprising, since both papers extend Heifetz, Meier, and Schipper (2006) by incorporating beliefs.

In more detail, $Y_{\mathcal{F},i} = S_{\mathcal{F}} \times \times_{j\neq i}\Omega_{\mathcal{F},j} = \Omega_{\mathcal{F},-i}$ and $\Omega_{\mathcal{F},j} = \bigsqcup_{\mathcal{E}\lhd\mathcal{F}} Q_{\mathcal{E},j}$. Also, $Q_{\mathcal{E},j}$ is homeomorphic to $\Delta\left(Y_{\mathcal{E},j}\right)$ via $t_{\mathcal{E},j}$, so we can illustrate the relationship between $Y$, $\Omega$ and $Q$ loosely in the following diagram.




If the functions $\tau_i$ and $g_i$ were added to the triangular diagram, they would map an object to another object that is three steps ahead in the direction of the arrows. To show this, the triangle is unpacked in the diagram below.




Heifetz, Meier, and Schipper (2007, 2011b) define projections between layers of the type space and impose conditions relating projections to type functions. To compare their model to the present paper, projections must be defined in type spaces with unawareness. For that, nonredundancy of types is needed, i.e. $t \neq t'$ must imply that $g_i(t) \neq g_i(t')$ for at least one $i \in I_0$. The universal type space satisfies nonredundancy by construction, for other type spaces in this subsection it is assumed.

Nonredundancy is equivalent to the condition that if $t \neq t'$, then there exists an event $F$ with $t \in F$ and $t' \notin F$. If $g_0(t) \neq g_0(t')$, then there is a $\sigma$-algebra $\mathcal{F}$ on $S$ and an $E_S \in \mathcal{F}$ such that $t \in E_S \times \times_{i\in I} M_{\mathcal{F},i}$ and $t' \notin E_S \times \times_{i\in I} M_{\mathcal{F},i}$. If $g_i(t) \neq g_i(t')$ for $i \neq 0$, then $g_i(t)(E) \geq r$ and $g_i(t')(E) < r$ for some $E \in \mathcal{F}$ and $r \in [0,1]$. This means $t \in B^i_r E$ and $t' \notin B^i_r E$.

The definition of projections starts from inverse projections of types. For $t \in \Omega_{\mathcal{E}}$ and any $\mathcal{F} \rhd \mathcal{E}$, define

$$(\rho^{\mathcal{F}}_{\mathcal{E}})^{-1} t = \{t' \in \Omega_{\mathcal{F}} : t \in E \Rightarrow t' \in E \text{ for all events } E\}.$$

Inverse projections commute because implications commute: if $t \in E \Rightarrow t' \in E$ and $t' \in E \Rightarrow t'' \in E$, then $t \in E \Rightarrow t'' \in E$.

If an event is not defined in $\Omega_{\mathcal{E}}$, then the implication in the definition trivially holds, so we can restate the definition using only events defined at $\Omega_{\mathcal{E}}$. For these events, the implication is actually an equivalence: if $t$ is in a layer where $E$ is defined, but not in $E$, then $t$ is in the negation of $E$. By the definition, $t' \in (\rho^{\mathcal{F}}_{\mathcal{E}})^{-1}t$ is then also in the negation of $E$. Therefore $t \notin E \Rightarrow t' \notin E$.

For all $t' \in (\rho^{\mathcal{F}}_{\mathcal{E}})^{-1}t$, let the projection of $t'$ into $\Omega_{\mathcal{E}}$ be $\rho^{\mathcal{F}}_{\mathcal{E}}t' = t$. It must be shown that this is a well-defined surjective measurable function. It is surjective because inverse projection was defined at every type.

By nonredundancy, $\rho^{\mathcal{F}}_{\mathcal{E}}$ is a function: for $t \neq t''$, there exists an event $E$ such that $t \in E$ and $t'' \in E^{c'}$. Since $E$ and $E^{c'}$ are disjoint and by definition $(\rho^{\mathcal{F}}_{\mathcal{E}})^{-1}t \in E$ and $(\rho^{\mathcal{F}}_{\mathcal{E}})^{-1}t'' \in E^{c'}$, the inverse projections of $t$ and $t''$ are disjoint. So there is no $t'$ that projects to more than one type in a lower awareness level.

To show measurability, take an event $E$ defined at $\Omega_{\mathcal{E}}$ and the inverse projection of its $\mathcal{E}$-section $(\rho^{\mathcal{F}}_{\mathcal{E}})^{-1}E_{\mathcal{E}} = \cup_{t \in E_{\mathcal{E}}} \{t' \in \Omega_{\mathcal{F}} : t \in F \Rightarrow t' \in F\}$. It will be shown that

$$(\rho^{\mathcal{F}}_{\mathcal{E}})^{-1}E_{\mathcal{E}} = E_{\mathcal{F}}, \tag{6}$$

which demonstrates that the inverse of $\rho^{\mathcal{F}}_{\mathcal{E}}$ takes measurable sets to measurable sets. If $t \in E_{\mathcal{E}}$, then by definition $(\rho^{\mathcal{F}}_{\mathcal{E}})^{-1}t \subseteq E_{\mathcal{F}}$, so $\cup_{t \in E_{\mathcal{E}}}(\rho^{\mathcal{F}}_{\mathcal{E}})^{-1}t \subseteq E_{\mathcal{F}}$. If on the other hand $t \notin E_{\mathcal{E}}$, then again by definition $(\rho^{\mathcal{F}}_{\mathcal{E}})^{-1}t \subseteq E^{c'}_{\mathcal{F}}$, so $(\rho^{\mathcal{F}}_{\mathcal{E}})^{-1}t \cap E_{\mathcal{F}} = \emptyset$ and therefore $\cup_{t \in E_{\mathcal{E}}}(\rho^{\mathcal{F}}_{\mathcal{E}})^{-1}t \supseteq E_{\mathcal{F}}$.

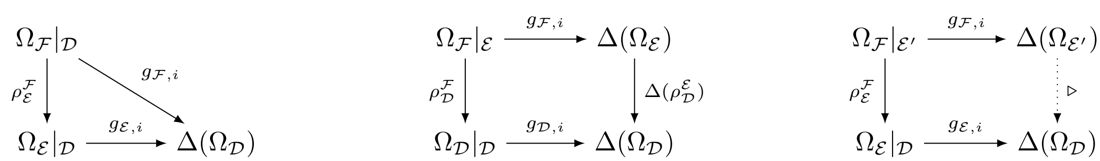
Projection is well-defined, because the previous paragraph implies $(\rho^{\mathcal{F}}_{\mathcal{E}})^{-1}\Omega_{\mathcal{E}} = \Omega_{\mathcal{F}}$, which means that for each $t' \in \Omega_{\mathcal{F}}$ there exists $t \in \Omega_{\mathcal{E}}$ such that $t' \in (\rho^{\mathcal{F}}_{\mathcal{E}})^{-1}t$.

An alternative definition of projection takes for $t' \in \Omega_{\mathcal{F}}$ and any $\mathcal{E} \triangleleft \mathcal{F}$ the intersection of all events containing $t'$ that are defined at $\Omega_{\mathcal{E}}$, formally $\rho^{\mathcal{F}}_{\mathcal{E}}t' = \cap_{t' \in E,\, \Omega(E) \triangleleft \mathcal{E}} E_{\mathcal{E}}$. The equivalence of this to the above definition based on inverse projections can be shown as follows. If $t'$ is in $F$ and $F$ is defined at $\Omega_{\mathcal{E}}$, then $\cap_{t' \in E,\, \Omega(E) \triangleleft \mathcal{E}} E_{\mathcal{E}} \subseteq F_{\mathcal{E}}$. This implication is actually an equivalence, because if $t'$ is not in $F$ and $F$ is defined at $\Omega_{\mathcal{E}}$, then $t' \in F^{c'}$, which implies $\cap_{t' \in E,\, \Omega(E) \triangleleft \mathcal{E}} E_{\mathcal{E}} \subseteq F^{c'}_{\mathcal{E}}$, i.e. $\cap_{t' \in E,\, \Omega(E) \triangleleft \mathcal{E}} E_{\mathcal{E}} \cap F_{\mathcal{E}} = \emptyset$. Therefore any event defined at $\Omega_{\mathcal{E}}$ contains $t'$ iff it contains its projection to $\Omega_{\mathcal{E}}$. This matches the first definition of projection.

With projections defined, the conditions imposed on the type space in Heifetz, Meier, and Schipper (2007) can finally be stated and shown to hold in the present paper. The conditions are

0) $g_i(t_{\mathcal{F}}) \in \Delta(\Omega_{\mathcal{E}})$ for some $\mathcal{E} \triangleleft \mathcal{F}$,
1) If $\mathcal{D} \triangleleft \mathcal{E} \triangleleft \mathcal{F}$, $t \in \Omega_{\mathcal{F}}$ and $g_i(t) \in \Delta(\Omega_{\mathcal{D}})$, then $g_i(\rho^{\mathcal{F}}_{\mathcal{E}}t)(\cdot) = g_i(t)(\cdot)$,
2) If $\mathcal{D} \triangleleft \mathcal{E} \triangleleft \mathcal{F}$, $t \in \Omega_{\mathcal{F}}$ and $g_i(t) \in \Delta(\Omega_{\mathcal{E}})$, then $g_i(\rho^{\mathcal{F}}_{\mathcal{D}}t)(\cdot) = g_i(t)\left((\rho^{\mathcal{E}}_{\mathcal{D}})^{-1}(\cdot)\right)$,
3) If $\mathcal{D} \triangleleft \mathcal{E} \triangleleft \mathcal{F}$, $t \in \Omega_{\mathcal{F}}$ and $g_i(\rho^{\mathcal{F}}_{\mathcal{E}}t) \in \Delta(\Omega_{\mathcal{D}})$, then $g_i(t) \in \Delta(\Omega_{\mathcal{E}'})$ for some $\mathcal{E}' \triangleright \mathcal{D}$.

Conditions 1, 2 and 3 require the following diagrams to commute. In the diagrams, $\Omega_{\mathcal{F}}|_{\mathcal{E}}$ denotes the subset of $\Omega_{\mathcal{F}}$ that $g_i$ maps to $\Omega_{\mathcal{E}}$. The function $\Delta(\rho^{\mathcal{E}}_{\mathcal{D}})$ maps $\mu_{\mathcal{E}} \in \Delta(\Omega_{\mathcal{E}})$ to $\mu_{\mathcal{D}} \in \Delta(\Omega_{\mathcal{D}})$ such that $\mu_{\mathcal{D}}(E) = \mu_{\mathcal{E}}((\rho^{\mathcal{E}}_{\mathcal{D}})^{-1}(E))$. The symbol $\triangleright$ at the dotted arrow means $\mathcal{E}' \triangleright \mathcal{D}$.




Condition 0 holds automatically by the definition of a type space with unawareness, while condition 1 is implied by 0, 2 and 3 according to Heifetz, Meier, and Schipper (2011b), so only the last two must be proved.

Let $g_i(\rho^{\mathcal{F}}_{\mathcal{E}}t) \in \Delta(\Omega_{\mathcal{D}})$ and $g_i(t) \in \Delta(\Omega_{\mathcal{E}'})$. If $\mathcal{E}' \triangleright \mathcal{D}$ fails, then $\rho^{\mathcal{F}}_{\mathcal{E}}t \in A^iE$ and $t \notin A^iE$ for some $E$ defined at $\Omega_{\mathcal{D}}$, but not at $\Omega_{\mathcal{E}'}$, so $\rho^{\mathcal{F}}_{\mathcal{E}}t$ fails the definition of projection. Therefore condition 3 must hold with the projections constructed above.

For condition 2, by definition of the projection operator, $g_i(t)(E_{\mathcal{E}}) = g_i(\rho^{\mathcal{F}}_{\mathcal{D}}t)(E_{\mathcal{D}'})$ for all events $E$ defined at both $\Omega_{\mathcal{E}}$ and $\Omega_{\mathcal{D}'}$, where $g_i(t) \in \Delta(\Omega_{\mathcal{E}})$ and $g_i(\rho^{\mathcal{F}}_{\mathcal{D}}t) \in \Delta(\Omega_{\mathcal{D}'})$. As shown in Eq. (6), $(\rho^{\mathcal{E}}_{\mathcal{D}'})^{-1}E_{\mathcal{D}'} = E_{\mathcal{E}}$, therefore $g_i(t)\left((\rho^{\mathcal{E}}_{\mathcal{D}'})^{-1}(\cdot)\right) = g_i(\rho^{\mathcal{F}}_{\mathcal{D}}t)(\cdot)$. To show that $\mathcal{D}' = \mathcal{D}$, take $E = \bigsqcup_{\mathcal{E}' \triangleright \mathcal{D}} E_{\mathcal{E}'}$, which implies $A^iE = \bigsqcup_{\mathcal{E}' \triangleright \mathcal{D}} (A^iE)_{\mathcal{E}'}$ and $t \in A^iE$. Then $\rho^{\mathcal{F}}_{\mathcal{D}}t$ is in $\cap_{t \in F_{\mathcal{F}},\, \Omega(F) \triangleleft \mathcal{F}} \subseteq A^iE$ by the second definition of projection. From this, $g_i(\rho^{\mathcal{F}}_{\mathcal{D}}t)$ is in $\Delta(\Omega_{\mathcal{D}})$, not in a strictly lower layer.

A similar proof works for condition 1. In this case, if $t \in \Omega_{\mathcal{F}}$ and $g_i(t) \in \Delta(\Omega_{\mathcal{D}})$, then in the universal type space there is a $t' \in \Omega_{\mathcal{E}}$ with $g_i(t) = g_i(t')$, so $t$ and $t'$ have the same awareness and belief. By definition of projection, $\rho^{\mathcal{F}}_{\mathcal{E}}$ must take $t$ to $t'$, since only in that case do $t$ and $t'$ belong to the same events.

The category of objects used in the current section is the same as in Heifetz, Meier, and Schipper (2011b). Due to this, the type functions and projections in both papers satisfy the same conditions. The similarity is driven by the common origin of the models—both modify Heifetz, Meier, and Schipper (2006).

## 7. Conclusion and potential extensions

This paper proved the existence of the universal type space with three kinds of unawareness, both in the purely measurable and in the topological case. The construction fixes the set of awareness levels, states of nature, and players. The universal space for all conceivable kinds of unawareness is unlikely to exist, due to similar cardinality considerations as in the possibility structures of Heifetz and Samet (1998a); Meier (2005). For the same reason, there is no universal type space that embeds type spaces over all possible spaces of states of nature or type spaces with all sets of players.

Unawareness of players was added to a set-based model, which appears to be new in the literature. Using several kinds of unawareness together also seems to be an innovation. One of these kinds of unawareness is a reinterpretation of the bounded reasoning in Kets (2010), so the present paper extends Kets (2010) to purely measurable spaces.

The most interesting extension of this paper would be to add quantification to the model, enabling the agents to express belief in the existence of unawareness without knowing exactly what they are unaware of. Quantification has been added to the logic of knowledge and awareness by Halpern and Rêgo (2009), so in the modal logic framework the universal type space with quantification and unawareness would combine one of these with Meier (2001) or Zhou (2009). In a set-based model quantification is difficult to express, since it is similar to conjunction of all events in a given class, but conjunction is a separate operator with a different interpretation. With an uncountable number of events, it is difficult to make arbitrary conjunctions of events measurable, so adding quantification to a model with probabilities is technically complex.

In games with unawareness, agents must believe in the existence of unawareness in order to generate novel predictions about behavior. Agents unaware of their unawareness act as if they were in a smaller game. Combining their behavior into a solution concept may be challenging, but from an agent's viewpoint the game is standard. For this reason, universal type spaces with self-aware unawareness, i.e. with quantification added to the model, are desirable.

Universal spaces of preference hierarchies along the lines of Epstein and Wang (1996); Bergemann, Morris, and Takahashi (2011); Gül and Pesendorfer (2010) can be constructed using the methods of this paper, since the preference spaces closely resemble belief spaces. A slight modification of the main theorem of Viglizzo (2005) can be used to prove the existence of a universal space for purely measurable utility functions, and an extension of Doberkat and Schubert (2011) will give the result in the Polish space framework.

Unawareness can be added to preference type spaces the same way as it has been added to universal belief type spaces in this paper. Preference spaces with unawareness provide the same foundation for games with unawareness as the preference spaces in the literature provide for standard games. Applications include axiomatizing solution concepts, distinguishing strategic generosity from true generosity and studying strategic distinguishability in games with unawareness.

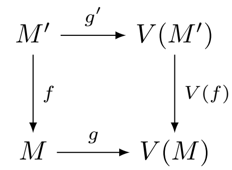
The universal type space with unawareness was constructed as a set-based model. An equivalent formulation would use modal logic, replacing events with formulas. The logic for probability and awareness would be a combination of the logic of awareness of Fagin and Halpern (1988) or Heifetz, Meier, and Schipper (2008) and the probability logic of Meier (2001) or Zhou (2009). Expressing the above results in a propositional notation would mostly be a matter of switching symbols.

## Appendix A. Proofs

To prove the existence of a universal type space with unawareness using Viglizzo (2005), the problem is restated in category-theoretic terms, as in Moss (2011). The following concepts from category theory are needed.

A category is a directed graph consisting of objects (nodes) and morphisms (arrows). In this paper, only the category $\mathtt{Meas}^{N\times I_0}$, consisting of vectors of measurable spaces as objects and vectors of measurable functions as morphisms, is needed. $I_0$ is the set of agents together with nature and $N$ is the cardinality of the set of awareness levels. A functor $V : \mathtt{Meas}^{N\times I_0} \to \mathtt{Meas}^{N\times I_0}$ maps each object $M \in \mathtt{Meas}^{N\times I_0}$ to an object $V(M) \in \mathtt{Meas}^{N\times I_0}$ and each morphism $f : M' \to M$ to a morphism $V(f) : V(M') \to V(M)$, preserving compositions of morphisms, so $V(f \circ g) = V(f) \circ V(g)$.

A coalgebra $(M, g)$ for a functor $V : \mathtt{Meas}^{N\times I_0} \to \mathtt{Meas}^{N\times I_0}$ is a vector of measurable spaces $M$ and a vector of measurable functions $g : M \to V(M)$. A coalgebra morphism from $(M', g')$ to $(M, g)$ is a morphism $f : M' \to M$ that satisfies $g \circ f = V(f) \circ g'$, as illustrated in the following diagram.




A final coalgebra for $V$ is one that has a unique coalgebra morphism from all coalgebras of $V$ into it. The type spaces of Heifetz and Samet (1998b) are coalgebras for the functor $W : \mathtt{Meas}^I \to \mathtt{Meas}^I$, defined as $W(M) = (\Delta(S \times M_{-i}))_{i \in I}$, and the universal type space is a final coalgebra of $W$ (Moss, 2011).

The next two lemmas characterize the type spaces with propositional unawareness as coalgebras of a certain functors in $\mathtt{Meas}^{N \times I_0}$. The proof of Theorem 1 follows.

**Lemma 12.** *A type space with propositional unawareness and awareness levels $\{\mathcal{F}^n : n \in N\}$ is a coalgebra for the functor $V : \mathtt{Meas}^{N \times I_0} \to \mathtt{Meas}^{N \times I_0}$ defined as $V = (V_{\mathcal{F},i})_{\mathcal{F} \lhd \mathcal{G}}^{i \in I_0}$, where for $i \in I$, $V_{\mathcal{F},i}(M) = \bigsqcup_{\mathcal{E} \lhd \mathcal{F}} \Delta(M_{\mathcal{E},-i})$ and $V_{\mathcal{F},0}(M) = S_{\mathcal{F}}$.*

*Proof.* By Definition 1, a type space with awareness is an (object, morphism) pair $(M, g)$ in $\mathtt{Meas}^{N \times I_0}$, where the morphism $g$ maps the object $M$ to another object $V(M) = \left(S_{\mathcal{F}}, (\bigsqcup_{\mathcal{E} \lhd \mathcal{F}} \Delta(M_{\mathcal{E},-i}))^{i \in I}\right)_{\mathcal{F} \lhd \mathcal{G}}$ in $\mathtt{Meas}^{N \times I_0}$. So $g$ maps $M$ to its image under the functor $V$. Therefore $(M, g)$ is a coalgebra for the functor. □

**Lemma 13.** *Coalgebra morphisms are type morphisms. Final coalgebras are universal type spaces.*

In the Polish space framework and without unawareness, this result is (Pintér, 2010, Corollary 2.8). The proof in the measurable space framework and with awareness does not differ significantly, but is presented below for completeness.

*Proof.* Coalgebra morphisms in $\mathtt{Meas}^{N \times I_0}$ and type morphisms are vectors of measurable functions. A coalgebra morphism $f : M' \to M$ for the functor $V$ satisfies $g \circ f = V(f) \circ g'$, meaning that for each $t' \in M'$, the same element of $V(M)$ results from applying $g \circ f$ as from $V(f) \circ g'$. This is exactly Eq. (3) in Definition 2.

By Lemma 12, type spaces are coalgebras for functors of the form $V(M) = \left(S_{\mathcal{F}}, (\bigsqcup_{\mathcal{E} \lhd \mathcal{F}} \Delta(M_{\mathcal{E},-i}))^{i \in I}\right)_{\mathcal{F} \lhd \mathcal{G}}$. The definition of a universal type space is that there exists a unique type morphism from any type space into it, and the definition of a final coalgebra is that there exists a unique coalgebra morphism from any coalgebra into it. Since type morphisms are coalgebra morphisms, universal type spaces are final coalgebras. □

*Proof of Theorem 1.* According to Viglizzo (2005, Theorem 7.1), any functor on $\mathtt{Meas}^I$ composed of $\Delta$, Cartesian products, disjoint unions, the identity, constant functors and projections from $\mathtt{Meas}^I$ to $\mathtt{Meas}$ has a final coalgebra. Viglizzo's proof remains unchanged if the functor is on $\mathtt{Meas}^{N \times I_0}$ and projections from $\mathtt{Meas}^{N \times I_0}$ to $\mathtt{Meas}$ are used, where $I_0$ and $N$ are countable. Since the functor in Lemma 12 satisfies the conditions, there exists a final coalgebra for it, which by Lemma 13 is the universal type space.

Suppose there are two final coalgebras $(\Omega, g)$ and $(\Omega', g')$. There is a unique morphism from any coalgebra to $(\Omega, g)$ and to $(\Omega', g')$, in particular the morphisms $f : \Omega \to \Omega'$, $f' : \Omega' \to \Omega$, $\mathrm{id}_\Omega : \Omega \to \Omega$ and $\mathrm{id}_{\Omega'} : \Omega' \to \Omega'$ are unique. Then $f' \circ f = \mathrm{id}_\Omega$ and $f \circ f' = \mathrm{id}_{\Omega'}$ by uniqueness, so $f = (f')^{-1}$ is an isomorphism. Any two final coalgebras are isomorphic.

By Viglizzo (2005, Theorem 2.3), the morphism $g$ in a final coalgebra is an isomorphism. This result in the Polish space framework is (Pintér, 2010, Corollary 2.9). □

Similarly to Lemma 12, it can be shown that type spaces with unawareness of agents, unawareness of higher-order reasoning and three kinds of unawareness are coalgebras of the following functors $V^a$, $V^b$ and $V^c$ respectively.

$$V^a(M) = \left( S, \left( \bigsqcup_{i \in J' \subseteq J} \Delta \left( M_0 \times \times_{j \in J' \setminus \{i\}} M_{J',j} \right) \right)^{i \in J}_{J \subseteq I} \right)$$

$$V^b(M) = \left( S, \left( \bigsqcup_{n \leq k-1} \Delta \left( M_0 \times \times_{j \in I \setminus \{i\}} M_{n,j} \right) \right)^{i \in I}_{1 \leq k \leq \infty} \right) \tag{7}$$

$$V^c(M) = \left( S_{\mathcal{F}}, \left( \bigsqcup_{\substack{\mathcal{E} \triangleleft \mathcal{F} \\ i \in J' \subseteq J \\ n \leq k-1}} \Delta \left( M_{\mathcal{E},0} \times \times_{j \in J' \setminus \{i\}} M_{\mathcal{E},J',n,j} \right) \right)^{i \in J \subseteq I}_{1 \leq k \leq \infty} \right)_{\mathcal{F} \triangleleft \mathcal{G}}$$

Repeating the proof Lemma 13, it is clear that type morphisms for the different kinds of unawareness considered in section 5 are coalgebra morphisms of the appropriate functors. The proof of Theorem 10 then follows along similar lines to Theorem 1.

*Proof of Theorem 10.* The functors $V^a$, $V^b$ and $V^c$ all satisfy the conditions of Theorem 7.1 of Viglizzo (2005), if the statement of that theorem is extended to countable products and disjoint unions. This can be done, keeping the same proof, since the operations are on measurable spaces. Therefore there exist final coalgebras for $V^a$, $V^b$ and $V^c$, which are the universal type spaces for unawareness of agents, unawareness of higher order reasoning, and three kinds of unawareness.

The uniqueness proof is the same as in Theorem 1. The morphisms in final coalgebras are isomorphisms, as argued in Theorem 1. □

*Proof of Theorem 11.* Theorem 3 of Schubert (2008) proves that for every functor in the class that contains the subprobability functor, the identity, and the constant functor for every Polish space and is closed under countable products, countable disjoint unions and compositions, there exists a final coalgebra. Section 5.6.1 of Doberkat and Schubert (2011) argues that the subprobability functor in Schubert (2008) can be replaced by the functor mapping each Polish space to the set of all Borel probability measures on it. The proof of Schubert can be extended to the category of vectors of measurable spaces, as in Theorem 7.1 of Viglizzo (2005).

The construction in Theorem 3 of Schubert (2008) relies on the Kolmogorov consistency theorem, restated in category-theoretic terms as Theorem 1 of Schubert (2008). It uses an inverse system of Polish measurable spaces with surjective measurable functions between them. For the purposes of this paper, the sequence of Polish measurable spaces can be the one obtained by repeatedly applying the functors in Lemma 12 or Eq. (7) to a singleton, and the surjective measurable functions can be the marginals. A consistent hierarchy of probability measures on the inverse system is a probability measure on the projective limit of the system, i.e. the hierarchy closes, encapsulating also the reasoning about the hierarchies. A consistent hierarchy of probability measures is one that satisfies the marginal condition familiar from the literature on universal type spaces (Mertens and Zamir, 1985; Brandenburger and Dekel, 1993). The proof of Theorem 3 of Schubert (2008) thus uses the same technique as these papers and extends their results.

Taking $\Delta$ to mean the set of all Borel probability measures, the functors in Lemma 12 and (7) fall into the class considered in Schubert (2008). These functors must therefore have final coalgebras. The final coalgebras are the universal type spaces by reasoning similar to Lemma 13.

The proofs that the universal type space is unique and that the morphism in it is an isomorphism are the same as in Theorem 1. □

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