

# Valemeid ja muid matemaatilisi tarkusi

mis mul elus ette on tulnud  
ja mida ma olen vajalikuks pidanud tallele panna

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# 1 Geomeetria

## 1.1 Koordinaatteisendused

### 1.1.1 Pinnaelement koordinaatteisendusel

Üleminekul koordinaatsüsteemilt  $(x, y)$  uutele koordinaatidele  $(u = u(x, y), v = v(x, y))$  teisenevad väikesed koordinaatide sihilised lõigud nii:

$$\begin{pmatrix} du \\ dv \end{pmatrix} = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} \begin{pmatrix} dx \\ dy \end{pmatrix}, \quad (1.1)$$

kus paremal poolel esimene on vastavate osatuletiste maatriks (jakobjaan  $\mathcal{J}$ ). Pinnaelement avaldub

$$dS = dx dy = \frac{du dv}{|\det \mathcal{J}|}, \quad (1.2)$$

kus  $|\det \mathcal{J}| = du dv \sin(\widehat{u, v})$  on kahe lühikese lõigu  $du$  ja  $dv$  poolt defineeritud elementaarpindala. Analoogiline seos kehtib ka kõrgemate mõõtmete puhul.

## 1.2 Kolmnurk

### 1.2.1 Seos kolmnurga külgede ja nurkade vahel

$$a^2 + b^2 - 2ab \cos \gamma = c^2. \quad (1.3)$$

Täisnurksel kolmnurgal, kui  $\gamma = 90^\circ$ , kehtib *Pythagorase teoreem*:

$$a^2 + b^2 = c^2. \quad (1.4)$$

### 1.2.2 Kolmnurga pindala

Kui kolmnurk on tasandil antud kolme punktiga  $(0, 0)$ ,  $(X_A, Y_A)$  ning  $(X_B, Y_B)$  siis kolmnurga pindala on

$$S = \frac{1}{2} |X_A \cdot Y_B - Y_A \cdot X_B| = \frac{1}{2} \text{abs} \begin{vmatrix} X_A & Y_A \\ X_B & Y_B \end{vmatrix}. \quad (1.5)$$

Tõestus: joonista kolmnurk välja ja vaata, millised pindalad kirjeldab determinant.

## 1.3 Muud

### 1.3.1 Joone kõverus

Parameetriselt antud joone kõverus:

$$k = \frac{x'y'' - y'x''}{(x'^2 + y'^2)^{3/2}} \quad (1.6)$$

## 2 Analüüs

### 2.1 Logaritm

Logaritmi omadused:

$$\log x^\alpha = \alpha \log x \quad \text{ning} \quad \log(xy) = \log x + \log y \quad (2.1)$$

### 2.2 Limits

$$\lim_{x \rightarrow \infty} \frac{1}{x} \frac{\phi(x)}{1 - \Phi(x)} = 1, \quad (2.2)$$

where  $\phi(\cdot)$  and  $\Phi(\cdot)$  are normal density and distribution functions.

*Proof.* The fraction can be written as

$$\frac{\phi(x)}{1 - \Phi(x)} = \frac{\phi(x)}{\int_x^\infty \phi(s) ds}.$$

The integral can be expressed as the Riemann limit

$$\int_x^\infty \phi(s) ds = \lim_{\delta \rightarrow 0} [\phi(x)\delta + \phi(x + \delta)\delta + \phi(x + 2\delta)\delta + \dots].$$

Using the expression for  $\phi(\cdot)$  we get

$$\phi(x + \delta) = \phi(x)e^{-x\delta}e^{-\delta^2/2}$$

and hence we may write the denominator in (2.2) as

$$x\phi(x) \lim_{\delta \rightarrow 0} [1 + e^{-x\delta}e^{-\delta^2/2} + e^{-2x\delta}e^{-4\delta^2/2} + e^{-3x\delta}e^{-9\delta^2/2} + \dots]\delta.$$

This expressions  $e^{-x\delta}$ ,  $e^{-2x\delta}$  and so on for a geometric sequence with sum  $1/(1 - e^{-x\delta}) \approx 1/x\delta$  as  $x\delta \rightarrow 0$ . Accordingly, we let  $\delta \rightarrow 0$  and  $x \rightarrow \infty$  in such a way that  $x\delta \rightarrow 0$ . We have to show that the other terms  $e^{n^2\delta^2/2}$  do not "disturb" the geometric sequence too much.

Now find  $n^*$ , starting of which the residual sum on the geometric sequence  $1 + e^{-x\delta} + e^{-2x\delta} + \dots$  is less than  $\varepsilon > 0$ :

$$\sum_{n=n^*}^{\infty} e^{-nx\delta} < \varepsilon \quad \Rightarrow \quad n^* > -\frac{\log \varepsilon + \log(1 - e^{-x\delta})}{x\delta}$$

choose  $n^{**} > -\frac{\log \varepsilon}{x\delta} + 1 > n^*$ . Now find

$$\exp \frac{n^{**2}\delta^2}{2} = \exp \left( \frac{\log^2 \varepsilon}{2x^2} + \delta \frac{\log \varepsilon}{x} + \frac{\delta^2}{2} \right).$$

Because all the terms in parenthesis converge to as  $x \rightarrow \infty$  and  $\delta \rightarrow 0$ , the exponent converges to 1. Hence, at the limit, the Riemann sum is solely determined by the geometric sequence, and we have denominator in (2.2) equal to  $\phi(x)$ .  $\square$

## 2.3 Differentiation

### 2.3.1 Tuletis mingis suunas (directional derivative)

$$f'(x; u) = \lim_{h \rightarrow 0} \frac{f(x + hu) - f(x)}{h} \quad (2.3)$$

### 2.3.2 Jakobiaan (Jacobian determinant)

$$\frac{\partial(f_1, \dots, f_n)}{\partial(x_1, \dots, x_n)} = \det Df(x) = \begin{vmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_2}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_1} \\ \frac{\partial f_1}{\partial x_2} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_1}{\partial x_n} & \frac{\partial f_2}{\partial x_n} & \cdots & \frac{\partial f_n}{\partial x_n} \end{vmatrix} \quad (2.4)$$

### 2.3.3 Ümbrikuteoreem (envelope theorem)

Olgu  $M$  defineeritud kui optimum funktsioonist  $f$ :

$$M(a) = \max_x f(x, a), \quad (2.5)$$

kus  $a$  on parameeter. Siis

$$\frac{dM(a)}{da} = \left. \frac{\partial f(x^*, a)}{\partial a} \right|_{x^*=x(a)}. \quad (2.6)$$

### 2.3.4 Elementaarfunktsioonide tuletised

$$\tan \phi' = \frac{1}{\cos^2 \phi} \quad (2.7)$$

$$\arctan x' = \frac{1}{1 + x^2} \quad (2.8)$$

### 2.3.5 Normal Density Related Derivatives

#### One-Dimensional Case

$$\frac{d}{dx} \Phi(-x) = \phi(x) \quad (2.9)$$

$$\frac{d}{dx} \phi\left(\frac{x - \mu}{\sigma}\right) = -\frac{1}{\sigma} \left(\frac{x - \mu}{\sigma}\right) \phi\left(\frac{x - \mu}{\sigma}\right) \quad (2.10)$$

**Multi-Dimensional Case** Let  $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ ,  $\Sigma = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}$  be the 2-dimensional variance-covariance matrix and  $\phi(\cdot, \cdot)$  the 2-D normal density, defined in (4.105):

$$\frac{\partial}{\partial x_1} \phi(x, \Sigma) = \phi(x, \Sigma) \frac{\rho x_2 - x_1}{1 - \rho^2} \quad (2.11)$$

$$\begin{aligned}\frac{\partial}{\partial \rho} \phi(x, \Sigma) &= \phi(x, \Sigma) \left[ \frac{\rho}{1 - \rho^2} (1 - x' \Sigma^{-1} x) + \frac{x_1 x_2}{1 - \rho^2} \right] = \\ &= \phi(x, \Sigma) \left[ \frac{\rho}{1 - \rho^2} - \frac{\rho}{(1 - \rho^2)^2} (x_1^2 - 2\rho x_1 x_2 + x_2^2) + \frac{x_1 x_2}{1 - \rho^2} \right] \quad (2.12)\end{aligned}$$

$$\frac{\partial^2}{\partial x_1 \partial x_2} \phi(x, \Sigma) = \phi(x, \Sigma) \left[ \frac{(x_1 - \rho x_2)(x_2 - \rho x_1)}{(1 - \rho^2)^2} + \frac{\rho}{1 - \rho^2} \right] \quad (2.13)$$

### 2.3.6 Derivatives of Gamma Function

$$\Gamma'(x) = \psi(x)\Gamma(x) \quad (2.14)$$

$$\Gamma''(x) = \Gamma(x) [\psi'(x) + \psi^2(x)], \quad (2.15)$$

kus  $\psi(x)$  on digamma funktsioon.

### 2.3.7 Üldised diferentseerimisreeglid

#### Pöördfunktsiooni tuletis

$$\frac{d}{dx} f^{-1}(x) = \frac{1}{\frac{d}{dx} f(x)} \Big|_{x=f^{-1}(y)} = \frac{1}{f'[f^{-1}(y)]} \quad (2.16)$$

**Mitmekihilise (liit-) funktsiooni tuletis** Olgu  $v = v(b)$  ja  $f(v) = f(v(b)) = g(b)$ . Tuletised:

$$\frac{\partial g}{\partial b} = \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial b} \equiv f'(v(b)) \cdot v'(b) \quad (2.17)$$

$$\frac{\partial^2 g}{\partial b^2} = \frac{\partial^2 f}{\partial v^2} \left( \frac{\partial v}{\partial b} \right)^2 + \frac{\partial f}{\partial v} \frac{\partial^2 v}{\partial b^2} \quad (2.18)$$

## 2.4 Integreerimine

### Leibnitzi reegel

$$\frac{\partial}{\partial x} \int_{a(x)}^{b(x)} f(y) dy = f[b(x)]b'(x) - f[a(x)]a'(x) \quad (2.19)$$

$$\begin{aligned}\frac{\partial}{\partial x} \int_{a(x)}^{b(x)} f(y, x) dy &= f[b(x), x]b'(x) - f[a(x), x]a'(x) + \\ &+ \int_{a(x)}^{b(x)} \frac{\partial}{\partial x} f(y, x) dy \quad (2.20)\end{aligned}$$

#### 2.4.1 Jaotusfunktsioonide integraalid

**Normaaljaotus** Tõestuseks tuleb kirjutada  $\phi(\cdot)$  lahti.

$$\int \phi^2(x) dx = \frac{1}{2\pi} \Phi(\sqrt{2}x) \quad (2.21)$$

$$\int x\phi(x) dx = -\phi(x) \quad (2.22)$$

$$\int_{-\infty}^{\infty} x\phi\left(\frac{x-\mu}{\sigma}\right) dx = \sigma\mu \quad (2.23)$$

$$\int_a^b x\phi\left(\frac{x-\mu}{\sigma}\right) dx = \sigma\mu \left[ \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right) \right] + \sigma^2 \left[ \phi\left(\frac{a-\mu}{\sigma}\right) - \phi\left(\frac{b-\mu}{\sigma}\right) \right] \quad (2.24)$$

$$\int x^2\phi(x) dx = -x\phi(x) + \Phi(x) \quad (2.25)$$

$$\int_a^b x^2\phi(x) dx = a\phi(a) - b\phi(b) + \Phi(b) - \Phi(a) \quad (2.26)$$

$$\int x^2\phi\left(\frac{x}{\sigma}\right) dx = -\sigma^2 x\phi\left(\frac{x}{\sigma}\right) + \sigma^3\Phi\left(\frac{x}{\sigma}\right) \quad (2.27)$$

$$\int_{-\infty}^{\infty} x^2\phi\left(\frac{x-\mu}{\sigma}\right) dx = \sigma(\mu^2 + \sigma^2) \quad (2.28)$$

$$\int_{-\infty}^{\infty} e^{ax}\phi\left(\frac{x-\mu}{\sigma}\right) dx = e^{a\mu + \frac{1}{2}\sigma^2 a^2} \quad (2.29)$$

$$\int_s^t \phi(u) \log \phi(u) du = \frac{1}{2} [\Phi(s) - \Phi(t)] (1 + \log 2\pi) + \frac{1}{2} [t\phi(t) - s\phi(s)] \quad (2.30)$$

Tõestus: integreeri ositi

$$\int_s^t \phi(u) \log \phi(u - \mu) du = \frac{1}{2} [\Phi(s) - \Phi(t)] (1 + \log 2\pi + \mu^2) + \frac{1}{2} [t\phi(t) - s\phi(s)] + \mu [\phi(s) - \phi(t)] \quad (2.31)$$

$$\int x\phi^2(x) dx \quad (2.32)$$

$$= -\frac{1}{2}\phi^2(x) \quad (2.33)$$

$$\int \phi(x)\Phi(x) dx \quad (2.34)$$

$$= \frac{1}{2}\Phi^2(x) \quad (2.35)$$

$$\int x\phi(x)\Phi(x) dx \quad (2.36)$$

$$= -\phi(x)\Phi(x) + \frac{1}{2\sqrt{\pi}}\Phi(\sqrt{2}x) \quad (2.37)$$

$$\int x^2\phi(x)\Phi(x) dx \quad (2.38)$$

$$= \frac{1}{2}\Phi^2(x) - x\phi(x)\Phi(x) - \frac{1}{2}\phi^2(x) \quad (2.39)$$

**Log-normal density** Let  $f(\cdot)$  be the log-normal density.

$$\begin{aligned} \int_a^\infty xf(x) dx &= \int_a^b \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{1}{2}\left(\frac{\log x - \mu}{\sigma}\right)^2\right] dx = \\ &= \frac{1 - \Phi\left(\frac{\log a - \mu - \sigma^2}{\sigma}\right)}{1 - \Phi\left(\frac{\log a - \mu}{\sigma}\right)} e^{\mu + \frac{1}{2}\sigma^2} \end{aligned} \quad (2.40)$$

$$\begin{aligned} \int_a^b xf(x) dx &= \int_a^b \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{1}{2}\left(\frac{\log x - \mu}{\sigma}\right)^2\right] dx = \\ &= e^{\mu + \frac{1}{2}\sigma^2} \left[ -\Phi\left(\frac{\log a - \mu - \sigma^2}{\sigma}\right) + \Phi\left(\frac{\log b - \mu - \sigma^2}{\sigma}\right) \right] \end{aligned} \quad (2.41)$$

#### 2.4.2 Other integrals

$$\int_0^t e^{-rT} dT = \frac{1}{r} [1 - e^{-rt}] \quad (2.42)$$

$$\int \frac{dx}{(p + qe^{ax})^2} = \frac{x}{p^2} + \frac{1}{ap(p + qe^{ax})} - \frac{1}{ap^2} \ln(p + qe^{ax}) \quad (2.43)$$

$$\int \log x dx = x \log x - x \quad (2.44)$$

$$\int e^{-ax} dx = \frac{1}{a} [1 - e^{-ax}] \quad (2.45)$$

$$\int xe^x dx = xe^x - e^x \quad (2.46)$$

$$\int xe^{-x} dx = -xe^{-x} - e^{-x} \quad (2.47)$$

$$\int xe^{ax} dx = \frac{x}{a} e^{ax} - \frac{1}{a^2} e^{ax} \quad (2.48)$$

$$\int_a^b xe^{ax} dx = \frac{1}{a} \left[ e^{ab} \left( b - \frac{1}{a} \right) - e^{aa} \left( a - \frac{1}{a} \right) \right] \quad (2.49)$$

$$\int x^2 e^{ax} dx = \frac{1}{a} x^2 e^{ax} - \frac{2}{a} \int xe^{ax} dx \quad (2.50)$$

$$\int_a^b x^2 e^{ax} dx = \frac{1}{a} \left[ e^{ab} \left( b^2 - \frac{2b}{a} + \frac{2}{a^2} \right) - e^{aa} \left( a^2 - \frac{2a}{a} + \frac{2}{a^2} \right) \right] \quad (2.51)$$

$$\int_0^\infty x^\alpha e^{-\beta x} dx = \frac{\Gamma(\alpha + 1)}{\beta^{\alpha+1}} \quad (2.52)$$

Let  $f(\cdot)$  be a distribution function and  $\bar{F}(\cdot)$  the corresponding survival function:

$$\int_c^b \left[ f(x) \int_c^x w(y) dy \right] dx = \int_c^b \bar{F}(x)w(x) dx \quad (2.53)$$

**Euleri konstant**

$$\int_0^{\infty} e^{-z} \log z dz = c \approx -0,5772 \quad (2.54)$$

### 2.4.3 Üldised integreerimise reeglid

**Muutuja vahetus integraali all** Olgu vaja üle minna muutujatelt  $(x_1, \dots, x_N)$  muutujatele  $(y_1, \dots, y_N)$ . Sel juhul

$$\begin{aligned} \int_V f(x_1, \dots, x_N) dx_1 \dots dx_N &= \\ &= \int_V f[y_1(x_1, \dots, x_N), \dots, y_N(x_1, \dots, x_N)] \frac{dy_1 \dots dy_N}{|\mathcal{J}|} = \\ &= \int_V g(y_1, \dots, y_N) \frac{dy_1 \dots dy_N}{|\mathcal{J}|}, \end{aligned} \quad (2.55)$$

kus

$$|\mathcal{J}| = \left| \frac{\partial(y_1, \dots, y_N)}{\partial(x_1, \dots, x_N)} \right| \quad (2.56)$$

on koordinaatseisenduse jakobjaani absoluutväärtus.

### 2.4.4 Analüütilised funktsioonid

**Gammafunktsioon**

$$\Gamma(p) = \int_0^{\infty} \lambda^{p-1} e^{-\lambda} d\lambda \quad (2.57)$$

Gammafunktsiooni omadus:

$$\Gamma(p+1) = p\Gamma(p) = p! \quad (2.58)$$

Tõestus: integreeri ositi.

**Digamma funktsioon**

$$\psi(\alpha) = \frac{d \log \Gamma(\alpha)}{d\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^{\infty} x^{\alpha-1} e^{-x} \log x dx. \quad (2.59)$$

Digamma omadus:

$$\psi(\alpha+1) = \frac{1}{\alpha} + \psi(\alpha). \quad (2.60)$$

Digamma arväärtused:

$$\psi(1) = -0,5772 \quad \psi(2) = 0,4228 \quad (2.61)$$

## Beetafunktsioon

$$B(p, q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} = \int_0^1 x^{p-1}(1-x)^{q-1} dx, \quad (2.62)$$

kusjuures  $p > 0$  ja  $q > 0$ .

### 2.4.5 Laplace'i teisendus

Laplace'i teisendus juhusliku muutuja  $X$  jaotusfunktsioonist on

$$L_f(s) = \mathbb{E} e^{-sX} = \int e^{-sx} dF_X(x). \quad (2.63)$$

Laplace'i teisendus on sama mis momendifunktsioon.

### 2.4.6 Numbriline integreerimine

**Monte-Carlo integraal** Olgu vaja leida

$$I = \int_a^b f(x) dx = (b-a) \int_a^b f(x) \frac{1}{b-a} dx = (b-a) \mathbb{E}[f(X)], \quad (2.64)$$

kus  $X \sim U(a, b)$ . Valimis suurusega  $N$  olgu  $x_1, \dots, x_N \sim i.i.d U(a, b)$ . Siis integraali hinnanguks on funktsiooni väärtuste keskmine ja veahinnanguks tema standardhälve valimis:

$$\hat{I}_N = (b-a) \frac{1}{N} \sum_{i=1}^N f(x_i) \quad (2.65)$$

$$\widehat{\text{Var}} \hat{I}_N = \frac{(b-a)^2}{N} \frac{1}{N} \sum_{i=1}^N \left[ f(x_i) - \frac{1}{N} \hat{I}_N \right]^2 \quad (2.66)$$

## 2.5 Optimisation

### 2.5.1 Second-order maximum/minimum conditions for constrained optimisation

The problem is

$$\begin{aligned} \max z &= f(x_1, x_2, \dots, x_n) \\ \text{s.t. } g(x_1, x_2, \dots, x_n) &= 0. \end{aligned} \quad (2.67)$$

Corresponding Lagrangian is

$$Z = f(x_1, x_2, \dots, x_n) - \lambda g(x_1, x_2, \dots, x_n). \quad (2.68)$$

Corresponding *bordered Hessian* is:

$$|\bar{H}| = \begin{vmatrix} 0 & g_1 & g_2 & \dots & g_n \\ g_1 & Z_{11} & Z_{12} & \dots & Z_{1n} \\ g_2 & Z_{21} & Z_{22} & \dots & Z_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ g_n & Z_{n1} & Z_{n2} & \dots & Z_{nn} \end{vmatrix} \quad (2.69)$$

and successive *principal minors* are:

$$|\bar{H}_2| = \begin{vmatrix} 0 & g_1 & g_2 \\ g_1 & Z_{11} & Z_{12} \\ g_2 & Z_{21} & Z_{22} \end{vmatrix} \quad |\bar{H}_3| = \begin{vmatrix} 0 & g_1 & g_2 & g_3 \\ g_1 & Z_{11} & Z_{12} & Z_{13} \\ g_2 & Z_{21} & Z_{22} & Z_{23} \\ g_3 & Z_{31} & Z_{32} & Z_{33} \end{vmatrix} \quad \dots \quad |\bar{H}_n| = |\bar{H}| \quad (2.70)$$

The second derivative  $d^2z$  is positive definite iff

$$\bar{H}_2 < 0, \quad \bar{H}_3 < 0, \quad \dots, \quad \bar{H}_n < 0,$$

and negative definite iff

$$\bar{H}_2 > 0, \quad \bar{H}_3 < 0, \quad \bar{H}_n > 0, \quad \dots$$

Note that  $\bar{H}_1$  is always negative.

Proof: Chiang (1984).

## 2.5.2 Optimal control (dynamic optimisation)

The problem is:

$$\begin{aligned} & \max_u \int_0^T F(t, y, u) dt \\ & \text{s.t. } \dot{y} = f(t, y, u) \\ & y(0) = A \quad y(T) \text{ free.} \end{aligned} \quad (2.71)$$

Corresponding *Hamiltonian* is

$$H(t, y, u, \lambda) \equiv F(t, y, u) + \lambda(t)f(t, y, u). \quad (2.72)$$

The first order conditions for optimum are

1.  $\max_u H(t, y, u, \lambda) \quad \forall t \in [0, T]$  or, less generally,  $\frac{\partial H}{\partial t} = 0$  (optimality condition).
2.  $\dot{y} = \frac{\partial H}{\partial \lambda}$  (equation of motion for  $y$ ).
3.  $\dot{\lambda} = -\frac{\partial H}{\partial y}$  (equation of motion for  $\lambda$ ).
4.  $\lambda(T) = 0$  (transversality condition).

Proof: Miller (1979)

## 2.5.3 Newton-Raphson maximization

Non-linear continuous function of  $N$ -dimensional parameter can, under suitable assumptions, be approximated as  $N$ -dimensional parabola. When running non-linear maximization, we may approximate the function in this way at the initial value of the parameter vector. The maximum of the approximation can be used as the initial value for the next step.

Let us maximise a function  $l(\vartheta)$  where  $\vartheta$  is a  $N$ -dimensional parameter vector. Let the initial value of the parameter be  $\vartheta_0$ . From Taylor's approximation:

$$l(\vartheta) \approx l(\vartheta_0) + \left. \frac{\partial l(\vartheta)}{\partial \vartheta} \right|_{\vartheta=\vartheta_0} (\vartheta - \vartheta_0) + \frac{1}{2} (\vartheta - \vartheta_0)' \left. \frac{\partial^2 l(\vartheta)}{\partial \vartheta \partial \vartheta'} \right|_{\vartheta=\vartheta_0} (\vartheta - \vartheta_0) \quad (2.73)$$

At the maximum  $\partial l(\vartheta)/\partial \vartheta = 0$  and hence the parameter value at the maximum (the initial value for the next iteration):

$$\vartheta_1 = \vartheta_0 - \left[ \left. \frac{\partial^2 l(\vartheta)}{\partial \vartheta \partial \vartheta'} \right|_{\vartheta=\vartheta_0} \right]^{-1} \left. \frac{\partial l(\vartheta)}{\partial \vartheta} \right|_{\vartheta=\vartheta_0} \quad (2.74)$$

The algorithm requires either programming the analytical Hessian matrix  $\frac{\partial^2 l(\vartheta)}{\partial \vartheta \partial \vartheta'}$ , or calculating the Hessian matrix by numeric differentiation. The first way may be complicated, the latter one slow and subject to numerical errors.

**BHHH maximization** BHHH is a particular version of Newton-Raphson algorithm, suitable for maximizing log-likelihood function only. BHHH uses the information equality for approximating the Hessian:

$$\mathbb{E} \left[ \frac{\partial^2 l(\boldsymbol{\vartheta})}{\partial \boldsymbol{\vartheta} \partial \boldsymbol{\vartheta}'} \right]_{\boldsymbol{\vartheta}=\boldsymbol{\vartheta}_0} = - \mathbb{E} \left[ \frac{\partial l(\boldsymbol{\vartheta})}{\partial \boldsymbol{\vartheta}'} \Big|_{\boldsymbol{\vartheta}=\boldsymbol{\vartheta}_0} \frac{\partial l(\boldsymbol{\vartheta})}{\partial \boldsymbol{\vartheta}} \Big|_{\boldsymbol{\vartheta}=\boldsymbol{\vartheta}_0} \right] \quad (2.75)$$

This algorithm does not require Hessian matrix (this is approximated). However, it typically requires around 10 times more iterations for convergence as the approximation may be quite imprecise when initial values are far off the target. Note also that while the estimates are exactly the same as in the case of NR algorithm, the standard errors may be different on a finite sample (Calzolari and Fiorentini, 1993).

### 3 Algebra

#### 3.1 Mõisted

**proper subset**  $A$  on  $B$  *proper subset* kui  $A \subseteq B$  kui  $B \not\subseteq A$ .

**proper subspace** Kui  $A$  ja  $B$  ja  $A$  on  $B$  *proper subset*. Näiteks tasandi tõeline (lineaarne) alamruum on sirge.

#### 3.2 Tehted hulkadega

$$\left(\bigcap_{A \in F} A\right)^C = \bigcup_{A \in F} A^C \quad (3.1)$$

$$\left(\bigcup_{A \in F} A\right)^C = \bigcap_{A \in F} A^C \quad (3.2)$$

$$A \setminus B = A \cap B^C \quad (3.3)$$

#### 3.3 Lihtsam algebra

**Newtoni binoomvalem**

$$(a + b)^n = \sum_{i=0}^n \binom{n}{i} a^{n-i} b^i. \quad (3.4)$$

$\binom{n}{i}$  on kombinatsioonide arv  $n$  elemendist  $i$  kaupa  $\binom{n}{i} \equiv C_n^i = \frac{n!}{(n-i)!i!}$ .

**Geomeetrilise jada summa**

$$S = 1 + q + q^2 + q^3 + \dots = \frac{1}{1 - q}. \quad (3.5)$$

Tõestus: kirjuta välja  $qS$ , lahuta ja avalda  $S$ . Märkus: kui jada on kujul  $S' = q + q^2 + q^3 + \dots$ , siis  $S' = qS$ . Oluline erijuht kui  $q = \frac{1}{1+r}$ :

$$S = 1 + \frac{1}{1+r} + \frac{1}{(1+r)^2} + \dots = 1 + \frac{1}{r}. \quad (3.6)$$

$$\sum_{i=0}^{\infty} ip^i = \frac{p}{(1-p)^2} \quad (3.7)$$

Tõestus: kui  $S$  on antud summa, siis avalda  $S - pS \dots$

**EkspONENT piirväärtusena**

$$\lim_{n \rightarrow \infty} \left(1 + \frac{a}{n}\right)^n = e^a. \quad (3.8)$$

Tõestus: arenda Newtoni binoomvalemiga ritta, arvesta (3.9).

### faktoriaalide jagatis

$$\lim_{n \rightarrow \infty} \frac{n!}{(n-m)!} = n^m \quad (3.9)$$

Märkus: siin on eeldatud, et  $m \not\rightarrow \infty$ . Jaga läbi, arvesta, et  $n-1 \approx n$ . Seose erijuht:

$$\lim_{n \rightarrow \infty} \frac{n}{m} = \lim_{n \rightarrow \infty} \frac{n!}{(n-m)!m!} = \frac{n^m}{m!}. \quad (3.10)$$

### 3.4 Taylori rida

**Taylori rida** Iga funktsiooni võib punkti  $x_0$  ümbruses esitada astmereana:

$$\begin{aligned} f(x) &= f(x_0) + f'(x_0)(x-x_0) + \frac{1}{2}f''(x_0)(x-x_0)^2 + \dots = \\ &= \sum_{i=0}^{\infty} f^{(i)}(x_0) \frac{(x-x_0)^i}{i!}. \end{aligned} \quad (3.11)$$

Tõestus: kirjuta samasugune astmerida tundmatute kordajatega välja, võrruta  $f(x-x_0)$ -ga ja võta järjest tuletisi. Taylori rea erijuht, kui  $x_0 = 0$  on McLaureni rida:

$$f(x) = f(0) + f'(0)x + \frac{1}{2}f''(0)x^2 + \dots = \sum_{i=0}^{\infty} f^{(i)}(0) \frac{x^i}{i!}. \quad (3.12)$$

### EkspONENTI astmerida

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \dots = \sum_{i=0}^{\infty} \frac{x^i}{i!}. \quad (3.13)$$

Tõestus: arenda  $e^x$  Taylori ritta.

### EkspONENTI piirväärtus

$$\lim_{x \rightarrow 0} e^x = 1 + x. \quad (3.14)$$

Tõestus: ekspONENTI astmereast. Märkus: piirväärtus  $1+x$  on kõige tavalisem, mida on vaja kasutada. Olenevalt ülesandest tuleb arvestada rohkem (või ka vähem) astmerea liikmeid.

### Logaritmi astmerida

$$\ln x = (x-1) - \frac{(x-1)^2}{2!} + \frac{(x-1)^3}{3!} - \frac{(x-1)^4}{4!} + \dots \quad (3.15)$$

Tõestus: arenda Taylori ritta  $x_0 = 1$  ümbruses.

### 3.5 Summade astmed

$$\left(\sum_{i=1}^N x_i\right)^2 = \sum_{i=1}^N x_i^2 + \sum_{\substack{i,j=1 \\ i \neq j}}^N x_i x_j \quad (3.16)$$

$$\left(\sum_{i=1}^N x_i\right)^3 = \sum_{i=1}^N x_i^3 + 3 \sum_{\substack{i,j=1 \\ i \neq j}}^N x_i x_j^2 + \sum_{\substack{i,j,k=1 \\ i \neq j; j \neq k; k \neq i}}^N x_i x_j x_k \quad (3.17)$$

$$\begin{aligned} \left(\sum_{i=1}^N x_i\right)^4 &= \sum_{i=1}^N x_i^4 + 4 \sum_{\substack{i,j=1 \\ i \neq j}}^N x_i x_j^3 + 3 \sum_{\substack{i,j=1 \\ i \neq j}}^N x_i^2 x_j^2 + 6 \sum_{\substack{i,j,k=1 \\ i \neq j; j \neq k; k \neq i}}^N x_i x_j x_k^2 + \\ &+ \sum_{\substack{i,j,k,l=1 \\ i,j,k,l \neq}}^N x_i x_j x_k x_l \end{aligned} \quad (3.18)$$

Ühekordsetes summades on  $N$  liiget, kahekordsetes  $N(N-1)$ , kolmekordsetes  $N(N-1)(N-2)$  ning neljakordsetes  $N(N-1)(N-2)(N-3)$ .

Tuletuskäik lähtub viimasel juhul niisugustest mõtetest:

- Kui komponentide indeksid ei tohi olla võrdsed, siis on järgmist komponenti võimalik valida ühe võrra vähem
- Esimesel liikmel võetakse sisse kõik komponendid, seega on  $C_0^4$  varianti.
- Teisel liikmel on kaks komponenti ( $x_i$  ja  $x_j$ ), ühte võetakse kolm korda, teist korra. Seega tuleb valida üks, mis iga kord välja jäetakse. Seega  $C_1^4$  võimalust.
- Kolmandal liikmel valitakse mõlemad komponendid kahe kaupa. Kokku on  $C_2^4 = 6$  võimalust kahe kaupa valida, kuna aga pole vahet kumb komponentidest on kumb, siis jääb järele pool nendest.
- Neljandal liikmel on kolm komponenti, ruutliikme valimiseks on  $C_2^4 = 6$  võimalust. Kuna teised liikmed on esimeses astmes, siis on küll vahe, kumbad me välja valime. Jääb 6.
- Viimane, kõiki üks kord,  $C_4^4 = 1$ .

Kui  $X \sim i.i.d$ , siis

$$\mathbb{E} \left( \sum_{i=1}^N x_i \right)^2 = N \mathbb{E} X^2 + N(N-1)(\mathbb{E} X)^2 = N^2(\mathbb{E} X)^2 + N \text{Var} X \quad (3.19)$$

### 3.6 Maatriksalgebra

$$A^{-1} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^{-1} = \frac{1}{|A|} \begin{bmatrix} a_{22} & -a_{21} \\ -a_{12} & a_{11} \end{bmatrix} \quad (3.20)$$

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} = \begin{bmatrix} E^{-1} & -E^{-1}BD^{-1} \\ -D^{-1}CE^{-1} & F^{-1} \end{bmatrix}, \quad \text{kus} \quad (3.21)$$

$$E = A - BD^{-1}C$$

$$E^{-1} = A^{-1} + A^{-1}BF^{-1}CA^{-1}$$

$$F = D - CA^{-1}B$$

$$F^{-1} = D^{-1}CE^{-1}BD^{-1}$$

(3.22)

### 3.7 Maatriksite ja vektorite diferentseerimine

Definition: let  $\lambda$  be a scalar and  $x$  a  $K \times 1$  vector. By definition

$$\frac{\partial \lambda}{\partial x'} = \begin{bmatrix} \frac{\partial}{\partial x_1} \lambda \\ \dots \\ \frac{\partial}{\partial x_k} \lambda \end{bmatrix} \quad (3.23)$$

More results:

$$\frac{\partial x}{\partial x'} = I \quad (3.24)$$

$$\frac{\partial \beta' x}{\partial \beta} = x \quad (3.25)$$

$$\frac{\partial Ax}{\partial x'} = A \quad \frac{\partial Ax}{\partial x} = A' \quad (3.26)$$

$$\frac{\partial x'x}{\partial x} = |x + x| = 2x \quad \frac{\partial x'Ax}{\partial x} = (A + A')x \quad (3.27)$$

Oluline: ühte korrutamist ei tohi teiseks muuta. Näiteks kui avaldis sisaldab nii maatrikskorrutist kui skalaariga korrutamist (skalaariga korrutamine on põhimõtteliselt sama mis Kroneckeri korrutis  $\otimes$ ), ei tohi endist skalaariga korrutamist diferentseerimise järel tõlgendada maatrikskorrutisena. Mis siis et skalaari asemel on nüüd maatriks:

$$\frac{\partial}{\partial \beta'} [(\beta'x) \otimes y] = \frac{\partial \beta'x}{\partial \beta'} \otimes y = x' \otimes y = yx'. \quad (3.28)$$

Skalaariga korrutamisel korrutatakse kõik maatriksi elemendid läbi sama skalaariga, seega päale tuletise võtmist tuleb kõik vektori  $y$  elemendid läbi korrutada tuletisvektoriga  $x'$ .

### 3.8 Võrratused

#### 3.8.1 Hölder's inequality

Let  $X$  and  $Y$  be random variables.

$$\mathbb{E}|XY| \leq \left\{ \mathbb{E} \left[ |X|^{\frac{1}{\alpha}} \right] \right\}^{\alpha} \left\{ \mathbb{E} \left[ |X|^{\frac{1}{1-\alpha}} \right] \right\}^{1-\alpha} \quad (3.29)$$

#### 3.8.2 Jenseni võrratus

$$Ef(x) < f(Ex), \quad (3.30)$$

Kui  $f(x)$  on kumer.

#### Kolmnurga võrratus

$$|x + y| \leq |x| + |y| \quad (3.31)$$

#### 3.8.3 Cauchy-Schwartzi võrratus

$$| \langle x, y \rangle | \leq \|x\| \cdot \|y\| \quad (3.32)$$

For sequences

$$\left( \sum a_i b_i \right)^2 \leq \left( \sum a_i^2 \right) \left( \sum b_i^2 \right). \quad (3.33)$$

For functions:

$$\langle x, y \rangle = \int x \cdot y \, dx \quad \|x\| = \sqrt{\langle x, x \rangle}$$

Vektorkujul:

$$(\mathbf{a} \cdot \mathbf{b})^2 \leq \|\mathbf{a}\|^2 \|\mathbf{b}\|^2 \quad (3.34)$$

ehk siis ka

$$\sum_i z_i z'_i \geq \frac{\sum_i a_i z_i \sum_i a_i z'_i}{\sum_i a_i^2} \quad (3.35)$$

#### 3.8.4 Inequalities, containing exponent

Proof in most cases by analysing the corresponding function.

$$e^a \geq a \quad (3.36)$$

$$1 - e^{-a} < a \quad (3.37)$$

$$(1 - e^{-a})e^{-a} < a \quad (3.38)$$

$$(1 + a)e^a \geq (1 + 2a) \quad (3.39)$$

$$(1 + a)e^{-a} < 1 \quad \text{if } a > 0 \quad (3.40)$$

$$(a - 1)e^a > -1 \quad \text{if } a > 0 \quad (3.41)$$

$$(1 + a^2)e^{-a} < 1 \quad \text{if } a > 0 \quad (3.42)$$

$$e^{-a} - e^{-b} < -a + b \quad \text{if } 0 < a < b \quad (3.43)$$

$$e^a - e^b = (a - b) + \frac{1}{2}(a^2 - b^2) + \frac{1}{6}(a^3 - b^3) + \dots \quad (3.44)$$

$$\geq (a - b) \quad \text{if } a \geq b \quad (3.45)$$

$$ae^{-b} - be^{-a} \geq a - b \quad \text{if } a \geq b \quad (3.46)$$

$$a^2e^b - b^2e^a \geq a^2 - b^2 + ab(a - b) + \frac{1}{6}a^2b^2(b - a) + \frac{1}{24}a^2b^2(b^2 - a^2) + \dots$$

if  $b \geq a$  (3.47)

## 4 Jaotused

### 4.1 Mõisted

#### 4.1.1 Stohhastiline domineerimine

Olgu  $F$  ja  $G$  jaotusfunktsioonid.  $G$  esimest järku domineerib  $F$ -i kui  $G(x) \leq F(x) \quad \forall x$  ehk  $\bar{G}(x) \geq \bar{F}(x) \forall x$ . Laias laastus annab  $G$  suuremad väärtused kui  $F$ .

#### 4.1.2 Sõltumatud juhuslikud muutujad

$X$  ja  $Y$  on sõltumatud  $\Leftrightarrow f(x, y) = f_X(x)f_Y(y) \Leftrightarrow F(x, y) = F_X(x)F_Y(y)$ .

#### 4.1.3 Expectations

Let support of random variable  $X$  be  $[a, b]$ . Expectation of  $X$

$$\mathbb{E} X = \int_a^b x dF_X(x) \quad (4.1)$$

$$= a + \int_a^b \bar{F}_X(x) dx. \quad (4.2)$$

#### Law of iterated expectations

$$\mathbb{E}_X [\mathbb{E}[Y|X]] = \mathbb{E}[Y] \quad (4.3)$$

## 4.2 Jaotustest üldiselt

### 4.2.1 Momendifunktsioon (*moment generating function*)

MGF avaldub ühemõõtmelisel juhul:

$$M_x(s) = \mathbb{E} e^{sx} = \int e^{sx} f(x) dx. \quad (4.4)$$

Momendifunktsiooni omadused:

$$M'_x(0) = \mathbb{E} x \quad M''_x(0) = \mathbb{E} x^2 \quad M_x^{(n)}(0) = \mathbb{E} x^n. \quad (4.5)$$

Kasulik asi on ka  $\log x$ -i momendifunktsioon:

$$M_{\log x}(s) = \mathbb{E} e^{s \log x} = \mathbb{E} x^s. \quad (4.6)$$

$N$ -mõõtmelisel juhul avaldub MGF:

$$\begin{aligned} M(s_1, s_2, \dots, s_N) &= \mathbb{E} e^{\sum_{i=1}^N s_i x_i} = \\ &= \int \dots \int e^{s_1 x_1} e^{s_2 x_2} \dots e^{s_N x_N} f(x_1, x_2, \dots, x_N) dx_1 dx_2 \dots dx_N. \end{aligned} \quad (4.7)$$

## 4.2.2 Kumulandifunktsoon (*cumulant-generating function*)

KGF avaldub MGF-i kaudu:

$$K_x(s) = \log M(s) \quad \text{või} \quad K(s_1, s_2, \dots, s_N) = \log M(s_1, s_2, \dots, s_N). \quad (4.8)$$

KGF-i omadus (ühemõõtmelisel juhul):

$$K'_x(0) = \mathbb{E} x \quad (4.9)$$

$$K''_x(0) = \text{Var } x \quad (4.10)$$

$$K'''_x(0) = \mathbb{E}(x - \mathbb{E}x)^3 \quad (4.11)$$

ja kahemõõtmelisel juhul:

$$\frac{\partial^2 K(0,0)}{\partial s_1 \partial s_2} = \text{Cov}(x_1, x_2). \quad (4.12)$$

## 4.2.3 Juhusliku muutuja funktsiooni jaotusfunktsioon

Olgu juhuslikud muutujad  $X$  ja  $Y$  kusjuures  $X = X(Y)$ . Siis

$$F_x(x) = \Pr[X < x] = \Pr[X < x(y)] = F_x[x(y)] = F_y(y) \quad (4.13)$$

ja

$$f_y(y) = F'_y(y) = \frac{d}{dy} F_x[x(y)] = \frac{d}{dx} F_x(x) \frac{dx}{dy} = f_x[x(y)] \frac{dx}{dy}. \quad (4.14)$$

## 4.3 Ühemõõtmelised diskreetsed jaotused

### 4.3.1 Bernoulli jaotus

Kõige lihtsam kulli-ja-kirja jaotus. Olgu sündmuse  $A$  tõenäosus  $p$ . Siis juhuslik muutuja

$$Y = \begin{cases} 1 & \text{kui } A, \\ 0 & \text{kui } \bar{A}. \end{cases} \quad (4.15)$$

on Bernoulli jaotusega, kusjuures  $\mathbb{E} Y = p$  ja  $\text{Var } Y = p(1-p)$ .

### 4.3.2 Binoomjaotus

On  $N$  ühesuguse sõltumatu Bernoulli jaotusega juhusliku muutuja summa jaotus. Olgu  $X = \sum^N Y_i$  kus  $Y_i$  on Bernoulli jaotusega parameetriga  $p$ . Siis

$$\Pr(X = x) = \binom{N}{x} p^x (1-p)^{N-x} \quad x \in \{0, \dots, N\} \quad (4.16)$$

$$EY = Np \quad (4.17)$$

$$E(Y - EY)^2 = Np(1-p) \quad (4.18)$$

$$E(Y - EY)^3 = Np(1-p)(1-2p) \quad (4.19)$$

$$E(Y - EY)^4 = Np(1-p)(-1+3p-3p^2) \quad (4.20)$$

Tõestus: ühe katse korral kirjuta lahti  $(y-p)^n$ , arvesta et  $Ey^n = p$  ( $i \neq 0$ ).  $N$  sõltumatu katse korral momendid liituvad.

### 4.3.3 Diskreetne jaotus

Jaotus kus juhuslikul muutujal võib olla lõplik hulk diskreetseid väärtusi.

### 4.3.4 Geomeetiline jaotus ( $Geo(p)$ )

Geomeetiline jaotus kirjeldab mingi hulga Bernoulli jaotusega suuruste järjest esinemist. Olgu sündmuse tõenäosus  $p$ . Tõenäosus, et järjest toimub  $n$  sündmust ja seejärel sündmuste jada katkeb on:

$$f(n) = (1 - p)p^n. \quad (4.21)$$

Jaotuse omadused:

$$\mathbb{E} n = \frac{1 - p}{p} \quad (4.22)$$

$$\text{Var } n = \frac{1 - p}{p^2} \quad (4.23)$$

### 4.3.5 Multinoomjaotus

Olgu üksikul katsel  $M$  võimalikku tulemust  $A_1 \dots A_M$  vastavate tõenäosustega  $p_1 \dots p_M$ , kusjuures  $\sum^M p_i = 1$ . Olgu  $N$ -katselises seerias  $N_i$  realiseerunud sündmuste  $A_i$  arv. Siis:

$$\mathbb{E} N_i = N p_i \quad (4.24)$$

$$\text{Var } N_i = N p_i (1 - p_i) \quad (4.25)$$

...

$$\text{Cov}(N_i, N_j) = -N p_i p_j \quad (4.26)$$

Tõestus: momentide arvutamisel võib multinoomjaotuse taandada binoomjaotuseks, kovariatsiooni jaoks kirjutada definitsioon lahti, arvesta et  $\mathbb{E} A_i A_j = 0$ .

### 4.3.6 Poissoni jaotus

Poissoni jaotusega on sõltumatute sündmuste arv ajaühikus. Kui ajaühikus toimub keskmiselt  $\lambda$  sündmust, siis tõenäosus, et toimub  $n$  sündmust aja  $t$  jooksul on:

$$f(n) \equiv p_p(n) = \frac{e^{-\lambda t} (\lambda t)^n}{n!} \quad (4.27)$$

$$P_p(n) \equiv \sum_{s=1}^n p_p(s) \quad (4.28)$$

Jaotuse omadused: jaotus on log-kumer,

$$\mathbb{E} n = \lambda t \quad \text{Var } n = \lambda t \quad (4.29)$$

Tõenäosus, et mingi aja  $t$  jooksul ei toimu ühtegi sündmust,  $f(0)$ , on eksponentsiaalse jaotusega  $\mathcal{E}(\lambda)$ .

ML-hinnang: Kui vaatluse  $i$  jooksul, mille kestus on  $t_i$  toimub  $n_i$  sündmust, siis kõigi vaatluste ML hinnang on:

$$\hat{\lambda} = \frac{\sum n_i}{\sum t_i} \quad \text{ja} \quad \text{Var } \hat{\lambda} = \frac{\sum n_i}{(\sum t_i)^2} \quad (4.30)$$

Poissoni summa tuletis aja järgi: Olgu

$$\vartheta(t) = \sum_{s=0}^{\infty} Q(s) \frac{(\lambda t)^s}{s!} e^{-\lambda t} = \mathbb{E}_s Q(s) \quad (4.31)$$

siis

$$\frac{\partial}{\partial t} \vartheta(t) = \lambda \sum_{s=0}^{\infty} [Q(s+1) - Q(s)] p_p(s) - \lambda Q(s) p_p(s) = \quad (4.32)$$

$$= \lambda \sum_{s=1}^{\infty} Q(s) [p_p(s-1) - p_p(s)] - \lambda Q(0) p_p(0) \quad (4.33)$$

$$\frac{\partial}{\partial t} P_p(s) = -\lambda p_p(s) \quad (4.34)$$

$$\frac{\partial}{\partial t} p_p(s) = \begin{cases} -\lambda p_p(s) & \text{kui } s = 0 \\ -\lambda p_p(s) + \lambda p_p(s-1) & \text{kui } s > 0 \end{cases} \quad (4.35)$$

## 4.4 Ühemõõtmelised pidevad jaotused

### 4.4.1 EkspONENTJAOTUS $\mathcal{E}(\theta)$

EkspONENTJAOTUS kirjeldab konstantse kiirusega hääbuvaid protsesse. Tõenäosustihedus:

$$f(t) = \theta e^{-\theta t}, \quad t \geq 0, \theta > 0 \quad (4.36)$$

jaotusfunktsioon:

$$F(t) = 1 - e^{-\theta t}, \quad (4.37)$$

momendifunktsioon:

$$M_T(s) = \frac{1}{1 - \frac{s}{\theta}}, \quad s < \theta. \quad (4.38)$$

Momendid:

$$\mathbb{E} T = \frac{1}{\theta} \quad (4.39)$$

$$\mathbb{E} T^2 = \frac{2}{\theta^2} \quad (4.40)$$

$$\text{Var}(T - \mathbb{E}(T))^2 = \frac{1}{\theta^2}. \quad (4.41)$$

$\log T$  on esimest liiki ekstreemväärtuste jaotusega.  $\log T$ -ga seotud suurused on:

$$M_{\log T}(s) = \frac{\Gamma(s+1)}{\theta^s} \quad (4.42)$$

$$K_{\log T}(s) = \log \Gamma(s+1) - s \log \theta \quad (4.43)$$

$$\mathbb{E} \log T = \psi(1) - \log \theta \quad (4.44)$$

$$\text{Var} \log T = \psi'(1), \quad (4.45)$$

kus  $\psi$  on digamma funktsioon.

Kui  $z_1 \sim \mathcal{E}(\theta_1)$  ja  $z_2 \sim \mathcal{E}(\theta_2)$  siis

$$\log z_1 - \log z_2 \sim \frac{\theta_1}{\theta_1 + \theta_2 e^{-x}} \sim \Lambda(x), \quad \text{kui } \theta_1 = \theta_2 \quad (4.46)$$

Tõestus: arvesta et  $\Pr(z_1/z_2 < \alpha) = \Pr(z_1 < \alpha z_2)$  ja integreeri.

### 4.4.2 Esimest liiki ekstreemväärtuste (log-weibulli) jaotus

(Type-1 extreme value distribution). Selle jaotusega on  $\log T$  kui  $T \sim \mathcal{E}(1)$ . Tihedusfunktsioon:

$$f(x) = e^x e^{-e^{-x}}, \quad (4.47)$$

jaotusfunktsioon:

$$F(x) = 1 - e^{-e^{-x}}. \quad (4.48)$$

Omadused:

$$\mathbb{E} x = -c \approx 0,5772 \quad (4.49)$$

Jaotuse maksimaalväärtus (mood) on kohal 0. Momendifunktsioon:

$$\psi(s) = \Gamma(1 + is) \quad (4.50)$$

#### 4.4.3 F-jaotus $F(n_1, n_2)$

F-jaotus tekib kahe  $\chi^2$  jaotusega suuruse jagamisel. Kui  $w_1 \sim \chi_{n_1}^2$  ja  $w_2 \sim \chi_{n_2}^2$  siis

$$\frac{\frac{w_1}{n_1}}{\frac{w_2}{n_2}} \sim F(n_1, n_2). \quad (4.51)$$

Tihedusfunktsioon:

$$f(x) = \frac{\left(\frac{n_1}{n_2}\right)^{\frac{n_1}{2}} x^{\frac{n_1}{2}-1}}{B\left(\frac{n_1}{2}, \frac{n_2}{2}\right) \left(1 + \frac{n_1}{n_2}x\right)^{\frac{n_1+n_2}{2}}} \quad (4.52)$$

#### 4.4.4 Gammajaotus $\mathcal{G}(\alpha, \beta)$

Kasutatakse heterogeensuse kirjeldamiseks kestusmodelites. Tõenäosustihe-  
dus:

$$f(x) = \frac{1}{\beta^\alpha} \frac{1}{\Gamma(\alpha)} x^{\alpha-1} e^{-\frac{x}{\beta}} \quad x > 0, \quad (4.53)$$

$\alpha > 0$  kirjeldab kuju ja  $\beta > 0$  on skaalaparameeter. Märkus: mõnikord kasu-  
tatakse ka parameetreid  $(\alpha, \frac{1}{\beta})$ .

Momendifunktsioon ja ootused:

$$M_x(s) = \frac{1}{(\beta s - 1)^\alpha} \quad (4.54)$$

$$K_x(s) = -\alpha \log(\beta s - 1) \quad (4.55)$$

$$\mathbb{E} x = \beta \alpha \quad (4.56)$$

$$\mathbb{E} x^2 = \beta^2 \alpha (\alpha + 1) \quad (4.57)$$

$$\text{Var } x = \beta^2 \alpha. \quad (4.58)$$

$\log x$  kirjeldav momendifunktsioon ja ootused:

$$M_{\log x}(s) = \frac{\Gamma(s + \alpha)}{\Gamma(\alpha)} \beta^s \quad (4.59)$$

$$K_{\log x}(s) = s \log \beta + \log \Gamma(s + \alpha) - \log \Gamma(\alpha) \quad (4.60)$$

$$\mathbb{E} \log x = \log \beta + \psi(\alpha) \quad (4.61)$$

$$\text{Var } \log x = \psi'(\alpha) \quad (4.62)$$

Erijuht on *normaalne gammajaotus*, mille keskväärtus on 1. Sel juhul

$$\alpha = \frac{1}{\beta} \equiv \eta \quad (4.63)$$

ja jaotusfunktsioon

$$f_x(x) = \eta^\eta \frac{1}{\Gamma(\eta)} x^{\eta-1} e^{-\eta x}. \quad (4.64)$$

Sel juhul:

$$\mathbb{E} x = 1 \quad (4.65)$$

$$\mathbb{E} x^2 = 1 + \frac{1}{\eta} \quad (4.66)$$

$$\text{Var } x = \frac{1}{\eta} \quad (4.67)$$

Teine oluline erijuht on  $\chi^2$ -jaotus. Kui  $\alpha = \frac{k}{2}$  ja  $\beta = 2$ , siis  $X$  jaotusfunktsioon on

$$f_x(x) = \frac{1}{2^{\frac{k}{2}} \Gamma(\frac{k}{2})} y^{\frac{k}{2}-1} e^{-\frac{y}{2}}. \quad (4.68)$$

Tolle kohta öeldakse  $\chi^2(k)$  jaotus.

#### 4.4.5 Hii-ruut jaotus $\chi^2(k)$

$\chi^2(k)$  jaotus tekib kui liita kokku  $k$  normaaljaotusega juhusliku suuruse ruutu. Jaotusfunktsioon:

$$f_x(x) = \frac{1}{2^{\frac{k}{2}} \Gamma(\frac{k}{2})} y^{\frac{k}{2}-1} e^{-\frac{y}{2}}. \quad (4.69)$$

#### 4.4.6 Log-normaalne jaotus $LN(\mu, \sigma^2)$

Log-normaalne jaotusega on juhuslik suurus, mille logaritm on normaaljaotusega. Tihedusfunktsioon:

$$f_x(x) = \frac{1}{\sqrt{2\pi\sigma x}} e^{-\frac{1}{2} \left[ \frac{\log x - \mu}{\sigma} \right]^2}. \quad (4.70)$$

Omadused:

$$\mathbb{E} x = e^{\mu + \frac{1}{2}\sigma^2} \quad (4.71)$$

$$\text{Var } x = e^{2\mu + \sigma^2} (e^{\sigma^2} - 1). \quad (4.72)$$

#### 4.4.7 Log-ühtlane jaotus

Kasutatakse palgajaotuse kirjeldamiseks. Tihedusfunktsioon:

$$f(x) = \frac{1}{x \log \beta - \log \alpha}, \quad 0 \leq \alpha \leq x \leq \beta < \infty. \quad (4.73)$$

#### 4.4.8 Logistiline jaotus

Jaotusfunktsioon, tihedusfunktsioon ja tolle tuletis:

$$\Lambda(x) = \frac{e^x}{1 + e^x} = \frac{1}{1 + e^{-x}} \quad (4.74)$$

$$f(x) = \frac{e^x}{(1 + e^x)^2} = \frac{e^{-x}}{(1 + e^{-x})^2} \quad (4.75)$$

$$f'(x) = e^{-x} \frac{e^{-x} - 1}{(e^{-x} + 1)^3} \quad (4.76)$$

Momendifunktsioon:

$$M(s) = \int \frac{e^{x(s+1)}}{(1 + e^x)^2} dx \quad (4.77)$$

#### 4.4.9 Normaaljaotus ehk Gaussi jaotus $N(\mu, \sigma^2)$

Normaaljaotusega on summa, mis tekib juhuslike suuruste liitmisel (tsentraalne piirteoreem). Jaotusfunktsioon ja tihedusfunktsioon:

$$\Phi(x) \quad \text{Ei ole analüütiliselt avaldatav} \quad (4.78)$$

$$\phi(x) = \frac{1}{\sqrt{2\pi}} \frac{1}{\sigma} e^{-\frac{1}{2} \frac{(x-\mu)^2}{\sigma^2}} \quad (4.79)$$

Momendifunktsioon:

$$M_x(s) = e^{\mu s + \frac{\sigma^2 s^2}{2}} \quad (4.80)$$

**Tinglikud ootused** Kui  $X \sim N(\mu, \sigma)$ , siis

$$\mathbb{E} X | X > a = \mu - \sigma \frac{\phi\left(\frac{a-\mu}{\sigma}\right)}{\Phi\left(\frac{a-\mu}{\sigma}\right)} = \mu - \sigma \lambda\left(\frac{a-\mu}{\sigma}\right) \quad (4.81)$$

$$\mathbb{E} X | X < a = \mu + \sigma \lambda\left(\frac{\mu-a}{\sigma}\right) \quad (4.82)$$

Kui  $X \sim N(0, \sigma)$ , siis

$$\mathbb{E}[X | X > a] = \sigma \lambda\left(-\frac{a}{\sigma}\right) \quad (4.83)$$

$$\text{Var}[X | X > a] = \sigma^2 \left[ 1 + a \lambda\left(-\frac{a}{\sigma}\right) - \lambda^2\left(-\frac{a}{\sigma}\right) \right] \quad (4.84)$$

$$\mathbb{E}[X | X < a] = -\sigma \lambda\left(\frac{a}{\sigma}\right) \quad (4.85)$$

$$\text{Var}[X | X < a] = \sigma^2 \left[ 1 - a \lambda\left(\frac{a}{\sigma}\right) - \lambda^2\left(\frac{a}{\sigma}\right) \right] \quad (4.86)$$

$$\mathbb{E}[X^2 | X < a] = \frac{-\sigma a \phi\left(\frac{a}{\sigma}\right) + \sigma^2 \Phi\left(\frac{a}{\sigma}\right)}{\Phi\left(\frac{a}{\sigma}\right)} \quad (4.87)$$

$$\mathbb{E}[X^2 | X > a] = \frac{\sigma a \phi\left(\frac{a}{\sigma}\right) + \sigma^2 [1 - \Phi\left(\frac{a}{\sigma}\right)]}{1 - \Phi\left(\frac{a}{\sigma}\right)} \quad (4.88)$$

$$\mathbb{E}[X^2 | X > -a \wedge X < a] = \sigma^2 - 2 \frac{\sigma a \phi\left(\frac{a}{\sigma}\right)}{1 - 2\Phi\left(\frac{a}{\sigma}\right)} \quad (4.89)$$

$$\mathbb{E}[X^2 | X < -a \vee X > a] = \mathbb{E}[X^2 | X > a] \quad (4.90)$$

#### 4.4.10 Pareto jaotus

Kasutatakse palgajaotuse ülemise poole kirjeldamisel. Jaotusfunktsioon:

$$F_x(x) = 1 - \left(\frac{x_0}{x}\right)^\alpha, \quad x \geq x_0 > 0; \quad \alpha > 0. \quad (4.91)$$

#### 4.4.11 Pööratud normaaljaotus

Tihedusfunktsioon:

$$f(t) = \frac{1}{t^{\frac{3}{2}}} \phi\left(\frac{\mu t - 1}{\sigma \sqrt{t}}\right) \quad (4.92)$$

ja jaotusfunktsioon:

$$F(t) = \Phi\left(\frac{\mu t - 1}{\sigma \sqrt{t}}\right) - e^{2\frac{\mu}{\sigma^2}} \Phi\left(-\frac{\mu t + 1}{\sigma \sqrt{t}}\right). \quad (4.93)$$

Kõik momendid on olemas kui  $\mu > 0$ :

$$\mathbb{E} T = \frac{1}{\mu} \quad (4.94)$$

$$\text{Var } T = \frac{\sigma^2}{\mu^3}. \quad (4.95)$$

Kui  $\mu = 0$ , on jaotus korralik, positiivsed momendid aga puuduvad.

#### 4.4.12 $t$ -Distribution

Used for  $t$ -test. For  $n$  degrees of freedom:

$$f(x) = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{n\pi}\Gamma\left(\frac{n}{2}\right)} \left(1 + \frac{x^2}{n}\right)^{-\frac{n+1}{2}} \quad (4.96)$$

$$\mathbb{E} X = 0 \quad (4.97)$$

$$\text{Var } X = \frac{n}{n-2} \quad (4.98)$$

$$\text{skewness } g_1 = 0 \quad (4.99)$$

$$\text{curtosis } g_2 = \frac{3n-6}{n-4} \quad (n > 4) \quad (4.100)$$

#### 4.4.13 Weibulli jaotus

Weibulli jaotus on eksponentjaotuse üldistus, kasutatakse ajas ühtlaselt kahaneva hasardi kirjeldamiseks. Omadused:

$$F(t) = 1 - e^{-(\lambda t)^\alpha} \quad (4.101)$$

$$f(t) = \alpha \lambda^\alpha t^{\alpha-1} e^{-(\lambda t)^\alpha} \quad (4.102)$$

$$\theta(t) = \alpha \lambda^\alpha t^{\alpha-1} \quad (4.103)$$

where  $\lambda$  is scale- and  $\alpha$  is the shape parameter.

#### 4.4.14 Ühtlane jaotus

Ühtlane jaotus on siis, kui juhusliku suuruse jaotus on kogu piirkonnas ühesugune.

## 4.5 Mitmemõõtmelised pidevad jaotused

### 4.5.1 Normaaljaotus $N(\mu, \Sigma)$

N-mõõtmelise normaajaotuse jaotusfunktsioon: Olgu

$$X \sim N(\mu, \Sigma), \quad (4.104)$$

kus  $\mu$  on keskvärtus ja  $\Sigma$  dispersioonimaatriks. Siis:

$$f_X(x) = (2\pi)^{-\frac{n}{2}} |\Sigma|^{-\frac{1}{2}} e^{-\frac{1}{2}(x-\mu)'\Sigma^{-1}(x-\mu)}. \quad (4.105)$$

**Normaaljaotuste korrutis** Kahe normaaljaotuse korrutise saab alati esitada kahe niisuguse normaaljaotuse korrutisena, millest teine ei sõltu ühest muutujast:  $\phi(x, y) = \phi(x|y)\phi(y) = \phi(y|x)\phi(x)$ .

$$\frac{1}{\sigma_1} \phi\left(\frac{x-ay}{\sigma_1}\right) \frac{1}{\sigma_2} \phi\left(\frac{y-b}{\sigma_2}\right) = \frac{1}{\sigma_x} \phi\left(\frac{x-ab}{\sigma_x}\right) \frac{1}{\sigma_y} \phi\left(\frac{y-\mu_y}{\sigma_y}\right), \quad (4.106)$$

kus

$$\sigma_x^2 = \sigma_1^2 + \sigma_2^2 a^2 \quad \sigma_y = \frac{\sigma_1 \sigma_2}{\sqrt{\sigma_1^2 + \sigma_2^2 a^2}}$$

$$\mu_y = \frac{\sigma_1^2 b + \sigma_2^2 a x}{\sigma_1^2 + \sigma_2^2 a^2}$$

The same in multi-dimensional case:

$$\begin{aligned} \frac{1}{\sigma_1} \phi\left(\frac{x_1-y}{\sigma_1}\right) \frac{1}{\sigma_1} \phi\left(\frac{x_2-y}{\sigma_1}\right) \dots \frac{1}{\sigma_1} \phi\left(\frac{x_n-y}{\sigma_1}\right) \frac{1}{\sigma_2} \phi\left(\frac{y}{\sigma_2}\right) &= \\ &= \prod_{i=1}^n \frac{1}{\sigma_1} \phi\left(\frac{x_i-y}{\sigma_1}\right) \frac{1}{\sigma_2} \phi\left(\frac{y}{\sigma_2}\right) = \\ &= \frac{1}{\sigma_x} \phi\left(\frac{x_1}{\sigma_x}\right) \frac{1}{\sigma_x} \phi\left(\frac{x_2}{\sigma_x}\right) \dots \frac{1}{\sigma_x} \phi\left(\frac{x_n}{\sigma_x}\right) \frac{1}{\sigma_y} \phi\left(\frac{y-\mu_y}{\sigma_y}\right) = \\ &= \prod_{i=1}^n \frac{1}{\sigma_x} \phi\left(\frac{x_i}{\sigma_x}\right) \frac{1}{\sigma_y} \phi\left(\frac{y-\mu_y}{\sigma_y}\right) \end{aligned} \quad (4.107)$$

where

$$\sigma_x^2 = \sigma_1^2 \frac{\sigma_1^2 + n\sigma_2^2}{\sigma_1^2 + (n-1)\sigma_2^2} \quad \sigma_y = \frac{\sigma_1 \sigma_2}{\sqrt{\sigma_1^2 + n\sigma_2^2}}$$

$$\mu_y = \frac{\sigma_2^2 \sum_{i=1}^n x_i}{\sigma_1^2 + n\sigma_2^2}$$

**Tinglikud ootused** Kui  $X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ , kus  $\boldsymbol{\Sigma} = \begin{bmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{bmatrix}$  ja  $\boldsymbol{\mu} = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}$ , siis

$$X_1|X_2 = x_2 \sim N\left(\mu_1 + \frac{\sigma_{12}}{\sigma_2^2}(x_2 - \mu_2), \sigma_1^2 - \frac{\sigma_{12}^2}{\sigma_2^2}\right). \quad (4.108)$$

(4.107 järeldus) ning

$$\mathbb{E}[X_1|X_2 > a] = \mu_1 + \frac{\sigma_{12}}{\sigma_2} \lambda\left(\frac{\mu_2 - a}{\sigma_2}\right) \quad (4.109)$$

$$\text{Var}[X_1|X_2 > a] = \sigma_{12}^2 [a\lambda(-a) - \lambda^2(-a)] + \sigma_1^2 \quad (4.110)$$

$$\mathbb{E}[X_1^2|X_2 \in \mathcal{A}] = \frac{\sigma_{12}^2}{\sigma_2^4} \mathbb{E}[X_2^2|X_2 \in \mathcal{A}] + \sigma_1^2 - \frac{\sigma_{12}^2}{\sigma_2^2} \quad (4.111)$$

Tõestus: kirjuta (4.108)  $\Rightarrow X_1 = \mu_1 + \frac{\sigma_{12}}{\sigma_2}(X_2 - \mu_2) + E$ , kus  $E$  ja  $U_2$  on sõltumatud (normaaljaotuse omadus). Leia  $X_1|X_2 \in \mathcal{A} = \frac{\sigma_{12}}{\sigma_2^2} \mathbb{E}[X_2|X_2 \in \mathcal{A}] + E$ .

Kui  $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \sim N(\mathbf{0}, \boldsymbol{\Sigma})$ , kus  $\boldsymbol{\Sigma} = \begin{bmatrix} 1 & \rho\sigma \\ \rho\sigma & \sigma^2 \end{bmatrix}$ , siis tihedus

$$f(u_2|u_1 > \alpha) = \frac{\Phi\left(\frac{-\alpha + \frac{\rho}{\sigma}u_2}{\sqrt{1 - \rho^2}}\right)}{\Phi(-\alpha)} \frac{1}{\sigma} \phi\left(\frac{u_2}{\sigma}\right). \quad (4.112)$$

## 4.6 Jaotuste pered

### 4.6.1 Stabiilne pere

Mittenegatiivsesse stabiilseesse perre kuuluvad jaotused, mille momendifunktsioon on

$$M_x(s) = e^{-s^\alpha}, \quad 0 < \alpha \leq 1. \quad (4.113)$$

Stabiilset pere omadused:

1. kui juhusliku muutuja  $X_i$  jaotusfunktsioon on  $G_\alpha$  mis kuulub stabiilseesse perre, siis juhusliku muutuja

$$Y = n^{-\frac{1}{\alpha}} \sum_{i=1}^N X_i \quad (4.114)$$

jaotusfunktsioon on kah  $G_\alpha$ .

2. Momendifunktsiooni tuletis

$$M'_x(s) = -\alpha s^{\alpha-1} M(s) \quad (4.115)$$

## 5 Estimators

### 5.1 Maximum likelihood

#### 5.1.1 Relationship with Kulbach-leibler distance

Let the random variables  $X_1, X_2, \dots, X_n \sim f(\cdot|\boldsymbol{\vartheta}), F(\cdot|\boldsymbol{\vartheta}), i.i.d.$  Let  $f(\cdot|\boldsymbol{\vartheta})$  be specified fully parametrically with a finite unknown parameter vector  $\boldsymbol{\vartheta}$ . The *log-likelihood* function is:

$$\ell(\boldsymbol{\vartheta}|X_1, X_2, \dots, X_n) = \frac{1}{n} \sum_{i=1}^n \log f(X_i|\boldsymbol{\vartheta}). \quad (5.1)$$

The *maximum likelihood* estimator of  $\boldsymbol{\vartheta}$  is the value of  $\boldsymbol{\vartheta}$  which maximises the log-likelihood function:

$$\hat{\boldsymbol{\vartheta}} = \arg \max_{\boldsymbol{\vartheta}} \ell(\boldsymbol{\vartheta}|X_1, X_2, \dots, X_n). \quad (5.2)$$

This can be written as

$$\hat{\boldsymbol{\vartheta}} = \arg \max_{\boldsymbol{\vartheta}} \int \log f(X_i|\boldsymbol{\vartheta}) dF_n(x), \quad (5.3)$$

where  $F_n(x)$  is the empirical distribution function:

$$F_n(x) = \frac{1}{n} \sum_{i=1}^n \mathbb{1}(X_i \leq x). \quad (5.4)$$

Further, we may write the estimator as

$$\hat{\boldsymbol{\vartheta}} = \arg \min_{\boldsymbol{\vartheta}} \left[ \int \log f_0(X_i) dF_n(x) - \int \log f(X_i|\boldsymbol{\vartheta}) dF_n(x) \right], \quad (5.5)$$

where  $f_0(\cdot)$  is the true density function of  $X$ . Further we may write

$$\hat{\boldsymbol{\vartheta}} = \arg \min_{\boldsymbol{\vartheta}} \int \log \frac{f_0(X_i)}{f_0(X_i|\boldsymbol{\vartheta})} dF_n(x) = KL(f_0, f|\boldsymbol{\vartheta}) \quad (5.6)$$

where  $KL(f, g)$  is the Kulbach-Leibler distance:

$$KL(f, g) = \mathbb{E}_f \left[ \log \frac{f(x)}{g(x)} \right]. \quad (5.7)$$

#### 5.1.2 Information matrix

Information matrix is defined as

$$I(\boldsymbol{\vartheta}) \equiv - \mathbb{E} \left[ \frac{\partial^2 \ell(\boldsymbol{\vartheta})}{\partial \boldsymbol{\vartheta} \partial \boldsymbol{\vartheta}'} \right] = \mathbb{E} \left[ \frac{\partial \ell(\boldsymbol{\vartheta})}{\partial \boldsymbol{\vartheta}} \frac{\partial \ell(\boldsymbol{\vartheta})}{\partial \boldsymbol{\vartheta}'} \right] \quad (5.8)$$

## 6 Stohhastilised protsessid

### 6.1 Autoregressiivsed (AR) protsessid

**AR(1) protsess** Juhuslik muutuja  $U$  järgib AR(1) protsessi kui  $U$  käesoleva perioodi realisatsioon on seotud eelmise perioodi omaga

$$u_t = \rho u_{t-1} + \varepsilon_t \quad (6.1)$$

ning  $\varepsilon_t$  väärtused eri ajaperioodidel on sõltumatud. Et protsess oleks stabiilne peab  $\rho$  väärtus jääma vahemikku  $(-1, 1)$ .

**AR(2) protsess** Juhuslik muutuja  $U$  järgib AR(2) protsessi kui  $U$  käesoleva perioodi realisatsioon on seotud kahe eelmise perioodi omaga

$$u_t = \rho_1 u_{t-1} + \rho_2 u_{t-2} + \varepsilon_t \quad (6.2)$$

ning  $\varepsilon_t$  väärtused eri ajaperioodidel on sõltumatud.

### 6.2 Hulkumine

Definitsioon: hulkumine (*random walk*) on statistiline protsess

$$z_{t+1} = z_t + \varepsilon_{t+1}. \quad (6.3)$$

#### 6.2.1 Hulkumine vastu barjääri

Olgu  $z_0 = 0$  ja  $\varepsilon \sim N(0, 1)$  *i.i.d.* protsess. Siis  $z_2$  jaotus tingimusel et  $z_1$  es ülela barjääri  $\alpha$  on

$$f(z_2|z_1 < \alpha) = \frac{1}{\sqrt{2}} \frac{\Phi\left(\sqrt{2}\alpha - \frac{1}{\sqrt{2}}z_2\right)}{\Phi(\alpha)} \phi\left(\frac{z_2}{\sqrt{2}}\right) \quad (6.4)$$

Tõestus: kirjuta  $\phi(x)$  lahti ja integreeri.

## 7 Statistilised mudelid

### 7.1 Tobit-2 model

Definition:

$$y_{1i}^* = z_i' \gamma + u_{1i} \quad (7.1)$$

$$y_{2i}^* = x_i' \beta + u_{2i} \quad (7.2)$$

$$y_{1i} = \begin{cases} 1, & \text{if } y_{1i}^* > 0 \\ 0, & \text{if } y_{1i}^* \leq 0. \end{cases} \quad (7.3)$$

$$y_{2i} = \begin{cases} y_{2i}^*, & \text{if } y_{1i}^* > 0 \\ 0, & \text{if } y_{1i}^* \leq 0. \end{cases} \quad (7.4)$$

Assume

$$\begin{pmatrix} U_1 \\ U_2 \end{pmatrix} \sim N \left( \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \rho \\ \rho & \sigma^2 \end{pmatrix} \right). \quad (7.5)$$

The Heckman two-step estimator in this case is as follows:  $\gamma$  can be consistently estimated with probit model. Further we may write:

$$\mathbb{E}[Y_2 | Y_1 > 0, \mathbf{x}, \mathbf{z}] = \mathbf{x}' \beta + \mathbb{E}[U_2 | U_1 > -z' \gamma] = \mathbf{x}' \beta + \rho \sigma \lambda(-z' \gamma) \quad (7.6)$$

$$\text{Var}[Y_2 | Y_1 > 0, \mathbf{x}, \mathbf{z}] = \mathbb{E}[U_2 | U_1 > -z' \gamma] = \sigma^2 + \rho^2 \sigma^2 [-z' \gamma \lambda(z' \gamma) - \lambda^2(z' \gamma)] \quad (7.7)$$

where  $\lambda(x) = \phi(x)/\Phi(x)$ ;  $\Phi(\cdot)$  and  $\phi(\cdot)$  are the normal cumulative distribution function and density function respectively.  $\rho$  and  $\sigma$  can be estimated regressing  $y_{2i}$  on  $x_i$  and  $\lambda(-z' \gamma)$ . From the coefficient of the latter,  $\beta_\lambda$  and the residual variance  $s^2$ , one can isolate  $\rho$  and  $\sigma$ :

$$\hat{\sigma}^2 = s^2 + \beta_\lambda^2 [\lambda^2(z' \gamma) - z' \gamma \lambda(z' \gamma)] \quad (7.8)$$

$$\hat{\rho} = \frac{\beta_\lambda}{\hat{\sigma}}. \quad (7.9)$$

Note that  $\hat{\rho}$  need not to be in  $[-1, 1]$ .

Denote:

$$r = \sqrt{1 - \rho^2} \quad (7.10)$$

$$u_{2i} = y_{2i} - x_i' \beta \quad (7.11)$$

$$B_i = \frac{z_i' \gamma + \frac{\rho}{\sigma} u_{2i}}{r} \quad (7.12)$$

$$C(B) = -\frac{\Phi(B)\phi(B)B + \phi(B)^2}{\Phi(B)^2} \quad (7.13)$$

The likelihood of the model is:

$$\ell = \sum_{i: y_{1i} \leq 0} \log \Phi(-z_i' \gamma) + \quad (7.14)$$

$$+ \sum_{i: y_{1i} > 0} \left[ \log \Phi(B_i) - \frac{1}{2} \log 2\pi - \log \sigma - \frac{1}{2} \frac{u_{2i}^2}{\sigma^2} \right]. \quad (7.15)$$

The gradient of the log-likelihood is:

$$\frac{\partial \ell}{\partial \gamma} = \sum_{i:y_{1i} \leq 0} -\lambda(z'_i \gamma) z_i + \sum_{i:y_{1i} > 0} \lambda(B_i) \frac{z_i}{r} \quad (7.16)$$

$$\frac{\partial \ell}{\partial \beta} = \sum_{i:y_{1i} > 0} \left[ \frac{u_{2i}}{\sigma^2} - \lambda(B_i) \frac{\rho}{\sigma} \frac{1}{r} \right] x_i \quad (7.17)$$

$$\frac{\partial \ell}{\partial \sigma} = \sum_{i:y_{1i} > 0} \left[ \frac{u_{2i}^2}{\sigma^3} - \frac{1}{\sigma} - \lambda(B_i) \frac{\rho}{\sigma^2} \frac{u_{2i}}{r} \right] \quad (7.18)$$

$$\frac{\partial \ell}{\partial \rho} = \sum_{i:y_{1i} > 0} \lambda(B_i) \frac{\frac{1}{\sigma} u_{2i} + \rho z'_i \gamma}{r^3}. \quad (7.19)$$

Hessian components are

$$\frac{\partial^2 \ell}{\partial \gamma \partial \gamma'} = - \sum_{i:y_{1i}=0} C(-z'_i \gamma) z_i z'_i + \sum_{i:y_{1i}=1} \frac{C(B)}{r} z_i z'_i \quad (7.20)$$

$$\frac{\partial^2 \ell}{\partial \gamma \partial \beta'} = - \sum_{i:y_{1i}=1} C(B) \frac{1}{\sigma} \frac{\rho}{r} z_i x'_i \quad (7.21)$$

$$\frac{\partial^2 \ell}{\partial \gamma \partial \sigma} = - \sum_{i:y_{1i}=1} C(B) \frac{\rho u_2}{\sigma^2 r^2} z_i \quad (7.22)$$

$$\frac{\partial^2 \ell}{\partial \gamma \partial \rho} = \sum_{i:y_{1i}=1} \left[ C(B) \frac{\frac{u_2}{\sigma} + \rho z'_i \gamma}{r^4} + \lambda(B) \frac{\rho}{r^3} \right] z_i \quad (7.23)$$

$$\frac{\partial^2 \ell}{\partial \beta \partial \beta'} = \sum_{i:y_{1i}=1} \frac{1}{\sigma^2} \left[ \frac{\rho^2}{r^2} C(B) - 1 \right] x_i x'_i \quad (7.24)$$

$$\frac{\partial^2 \ell}{\partial \beta \partial \sigma} = \sum_{i:y_{1i}=1} \left[ C(B) \frac{\rho^2}{\sigma^3} \frac{u_2}{r^2} + \frac{\rho}{\sigma^2} \frac{\lambda(B)}{r} - 2 \frac{u_2}{\sigma^3} \right] x_i \quad (7.25)$$

$$\frac{\partial^2 \ell}{\partial \beta \partial \rho} = \sum_{i:y_{1i}=1} \left[ -C(B) \frac{\frac{u_2}{\sigma} + \rho z'_i \gamma}{r^4} \frac{\rho}{\sigma} - \frac{\lambda(B)}{\sigma} \frac{1}{r^3} \right] x_i \quad (7.26)$$

$$\frac{\partial^2 \ell}{\partial \sigma^2} = \sum_{i:y_{1i}=1} \left[ \frac{1}{\sigma^2} - 3 \frac{u_2^2}{\sigma^4} + 2 \lambda(B) \frac{u_2}{r} \frac{\rho}{\sigma^3} + \frac{\rho^2}{\sigma^4} \frac{u_2^2}{r^2} C(B) \right] \quad (7.27)$$

$$\frac{\partial^2 \ell}{\partial \sigma \partial \rho} = - \frac{1}{r^3} \sum_{i:y_{1i}=1} \frac{u_2}{\sigma^2} \left[ C(B) \frac{\rho \left( \frac{u_2}{\sigma} + \rho z'_i \gamma \right)}{r} + \lambda(B) \right] \quad (7.28)$$

$$\frac{\partial^2 \ell}{\partial \rho^2} = \sum_{i:y_{1i}=1} \left[ C(B) \left( \frac{\frac{u_2}{\sigma} + \rho z'_i \gamma}{r^3} \right)^2 + \lambda(B) \frac{z'_i \gamma (1 + 2\rho^2) + 3\rho \frac{u_2}{\sigma}}{r^5} \right] \quad (7.29)$$

## 7.2 Tobit-2 Model with Binary Outcome

The underlying latent model:

$$y_i^{S*} = \beta^{S'} x_i^S + \epsilon_i^S \quad (7.30)$$

$$y_i^{O*} = \beta^{O'} x_i^O + \epsilon_i^O \quad (7.31)$$

$$y_i^S = \begin{cases} 1, & \text{if } y_i^{S*} > 0 \\ 0, & \text{if } y_i^{S*} \leq 0. \end{cases} \quad (7.32)$$

$$y_i^O = \begin{cases} \text{undetermined,} & \text{if } y_i^S = 0 \quad (\text{case 1}) \\ 0, & \text{if } y_i^{O*} \leq 0 \quad \text{and } y_i^S = 1 \quad (\text{case 2}) \\ 1, & \text{if } y_i^{O*} > 0 \quad \text{and } y_i^S = 1 \quad (\text{case 3}) \end{cases} \quad (7.33)$$

Assume

$$\begin{pmatrix} \epsilon_1 \\ \epsilon_2 \end{pmatrix} \sim N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}\right). \quad (7.34)$$

The log-likelihood function contains 3 components, corresponding to the cases in (7.33):

$$\ell = \sum_{i \in \text{case 1}} \log \Phi(-\beta^{S'} x_i^S) \quad (7.35)$$

$$+ \sum_{i \in \text{case 2}} \log \left[ 1 - \Phi(-\beta^{S'} x_i^S) - \bar{\Phi}_2\left(\begin{pmatrix} -\beta^{S'} x_i^S \\ -\beta^{O'} x_i^O \end{pmatrix}, \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}\right) \right] \quad (7.36)$$

$$+ \sum_{i \in \text{case 3}} \log \bar{\Phi}_2\left(\begin{pmatrix} -\beta^{S'} x_i^S \\ -\beta^{O'} x_i^O \end{pmatrix}, \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}\right), \quad (7.37)$$

where  $\bar{\Phi}_2(\cdot, \cdot)$  is the upper tail probability of 2-dimensional normal distribution.

Denote by  $\mathcal{L}_i$  the corresponding individual likelihood value. The score vector:

$$\begin{aligned} \frac{\partial}{\partial \beta^S} \ell &= \sum_{i \in \text{case 1}} \frac{1}{\mathcal{L}_i} \phi(-\beta^{S'} x_i^S) x_i^S \\ &+ \sum_{i \in \text{case 2}} \frac{1}{\mathcal{L}_i} \phi(\beta^{S'} x_i^S) \bar{\Phi}\left(\frac{\beta^{O'} x_i^O - \rho \beta^{S'} x_i^S}{\sqrt{1 - \rho^2}}\right) x_i^S \\ &+ \sum_{i \in \text{case 3}} \frac{1}{\mathcal{L}_i} \phi(\beta^{S'} x_i^S) \Phi\left(\frac{\beta^{O'} x_i^O - \rho \beta^{S'} x_i^S}{\sqrt{1 - \rho^2}}\right) x_i^S \\ \frac{\partial}{\partial \beta^O} \ell &= \sum_{i \in \text{case 2}} \frac{1}{\mathcal{L}_i} \phi(\beta^{O'} x_i^O) \bar{\Phi}\left(\frac{\beta^{S'} x_i^S - \rho \beta^{O'} x_i^O}{\sqrt{1 - \rho^2}}\right) x_i^O \\ &+ \sum_{i \in \text{case 3}} \frac{1}{\mathcal{L}_i} \phi(\beta^{O'} x_i^O) \Phi\left(\frac{\beta^{S'} x_i^S - \rho \beta^{O'} x_i^O}{\sqrt{1 - \rho^2}}\right) x_i^O \\ \frac{\partial}{\partial \rho} \ell &= - \sum_{i \in \text{case 2}} \phi_2\left(\begin{pmatrix} -\beta^{S'} x_i^S \\ -\beta^{O'} x_i^O \end{pmatrix}, \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}\right) + \sum_{i \in \text{case 3}} \phi_2\left(\begin{pmatrix} -\beta^{S'} x_i^S \\ -\beta^{O'} x_i^O \end{pmatrix}, \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}\right), \end{aligned}$$

where  $\phi_2(\cdot, \cdot)$  is 2-dimensional normal density.

### 7.3 Tobit-5 Model

Definitsioon (lühiduse mõttes on indeks  $i$  ära jäetud):

$$y_1^* = \mathbf{Z}'\boldsymbol{\gamma} + u_1 \quad (7.38)$$

$$y_2^* = \mathbf{X}'\boldsymbol{\beta}_2 + u_2 \quad (7.39)$$

$$y_3^* = \mathbf{X}'\boldsymbol{\beta}_3 + u_3 \quad (7.40)$$

$$y_2 = \begin{cases} y_2^* & \text{kui } y_1^* \leq 0 \\ 0 & \text{kui } y_1^* > 0 \end{cases} \quad (7.41)$$

$$y_3 = \begin{cases} y_3^* & \text{kui } y_1^* > 0 \\ 0 & \text{kui } y_1^* \leq 0 \end{cases} \quad (7.42)$$

Eeldatakse et jääkliikmete jaotus on niisugune:

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \sim N \left( \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \varrho_2\sigma_2 & \varrho_3\sigma_3 \\ \varrho_2\sigma_2 & \sigma_2^2 & \sigma_{23} \\ \varrho_3\sigma_3 & \sigma_{23} & \sigma_3^2 \end{pmatrix} \right). \quad (7.43)$$

#### 7.3.1 Heckmani kahesammuline hinnang

$\hat{\gamma}$  leitakse probiti abil. Edasi võib kirjutada

$$\begin{aligned} y_2 &= \mathbf{X}'\boldsymbol{\beta}_2 - \varrho_2\sigma_2\lambda(-\mathbf{Z}'\boldsymbol{\gamma}) + e_2 \\ y_3 &= \mathbf{X}'\boldsymbol{\beta}_3 + \varrho_3\sigma_3\lambda(\mathbf{Z}'\boldsymbol{\gamma}) + e_3 \end{aligned} \quad (7.44)$$

Kusjuures

$$\begin{aligned} \sigma_{e_2} &= \sigma_2^2 \left\{ 1 - \varrho_2^2 \left[ \lambda^2(-\mathbf{Z}'\boldsymbol{\gamma}) - \mathbf{Z}'\boldsymbol{\gamma}\lambda(-\mathbf{Z}'\boldsymbol{\gamma}) \right] \right\} \\ \sigma_{e_3} &= \sigma_3^2 \left\{ 1 - \varrho_3^2 \left[ \lambda^2(\mathbf{Z}'\boldsymbol{\gamma}) + \mathbf{Z}'\boldsymbol{\gamma}\lambda(\mathbf{Z}'\boldsymbol{\gamma}) \right] \right\} \end{aligned} \quad (7.45)$$

Kui lähendada seost (7.44) OLS-ga, siis saab  $\lambda$  koefitsendi ja dispersiooni hinnangu abil leida  $\hat{\varrho}$  ja  $\hat{\sigma}$ . Märkus:  $\hat{\varrho}$  ei pruugi olla -1 ja 1 vahel.

#### 7.3.2 Maksimum-laiklikhuud hinnang

Mudeli log-laiklikhuud on:

$$\begin{aligned} l &= -\frac{N}{2} \log 2\pi + \\ &+ \sum_2 \left\{ -\log \sigma_2 - \frac{1}{2} \left( \frac{u_2}{\sigma_2} \right)^2 + \log \Phi \left[ -\frac{\mathbf{Z}'\boldsymbol{\gamma} + \frac{\varrho_2}{\sigma_2} (y_2 - \mathbf{X}'_i\boldsymbol{\beta}_2)}{\sqrt{1 - \varrho_2^2}} \right] \right\} \\ &+ \sum_3 \left\{ -\log \sigma_3 - \frac{1}{2} \left( \frac{y_3 - \mathbf{X}'_i\boldsymbol{\beta}_3}{\sigma_3} \right)^2 + \log \Phi \left[ \frac{\mathbf{Z}'\boldsymbol{\gamma} + \frac{\varrho_3}{\sigma_3} (y_3 - \mathbf{X}'_i\boldsymbol{\beta}_3)}{\sqrt{1 - \varrho_3^2}} \right] \right\}. \end{aligned} \quad (7.46)$$

Valikud 2 ja 3 erinevad ainult avaldise märgi poolest funktsiooni  $\Phi$  sees. Tuletised on:

$$\frac{\partial l}{\partial \gamma} = -\sum_2 \frac{\phi(B_2)}{\Phi(B_2)} \frac{\mathbf{Z}}{\sqrt{1-\varrho_2^2}} + \sum_3 \frac{\phi(B_3)}{\Phi(B_3)} \frac{\mathbf{Z}}{\sqrt{1-\varrho_3^2}} \quad (7.47)$$

$$\frac{\partial l}{\partial \beta_2} = \sum_2 \left[ \frac{\phi(B_2)}{\Phi(B_2)} \left( \frac{\varrho_2}{\sigma_2} \frac{\mathbf{X}}{\sqrt{1-\varrho_2^2}} \right) + \frac{u_2}{\sigma_2^2} \mathbf{X} \right] \quad (7.48)$$

$$\frac{\partial l}{\partial \sigma_2} = \sum_2 \left[ -\frac{1}{\sigma_2} + \frac{(y_2 - \mathbf{X}'\beta_2)^2}{\sigma_2^3} + \frac{\phi(B_2)}{\Phi(B_2)} \frac{\varrho_2}{\sigma_2^2} \frac{y_2 - \mathbf{X}'\beta_2}{\sqrt{1-\varrho_2^2}} \right] \quad (7.49)$$

$$\frac{\partial l}{\partial \varrho_2} = -\sum_2 \frac{\phi(B_2)}{\Phi(B_2)} \frac{\frac{1}{\sigma_2} (y_2 - \mathbf{X}'\beta_2) + \varrho_2 \mathbf{Z}'\gamma}{(1-\varrho_2^2)^{\frac{3}{2}}} \quad (7.50)$$

$$\frac{\partial l}{\partial \beta_3} = \sum_3 \left[ -\frac{\phi(B_3)}{\Phi(B_3)} \left( \frac{\varrho_3}{\sigma_3} \frac{\mathbf{X}}{\sqrt{1-\varrho_3^2}} \right) + \frac{u_3}{\sigma_3^2} \mathbf{X} \right] \quad (7.51)$$

$$\frac{\partial l}{\partial \sigma_3} = \sum_3 \left[ -\frac{1}{\sigma_3} + \frac{(y_3 - \mathbf{X}'\beta_3)^2}{\sigma_3^3} - \frac{\phi(B_3)}{\Phi(B_3)} \frac{\varrho_3}{\sigma_3^2} \frac{y_3 - \mathbf{X}'\beta_3}{\sqrt{1-\varrho_3^2}} \right] \quad (7.52)$$

$$\frac{\partial l}{\partial \varrho_3} = \sum_3 \frac{\phi(B_3)}{\Phi(B_3)} \frac{\frac{1}{\sigma_3} (y_3 - \mathbf{X}'\beta_3) + \varrho_3 \mathbf{Z}'\gamma}{(1-\varrho_3^2)^{\frac{3}{2}}} \quad (7.53)$$

Teised tuletised:

$$\frac{\partial^2 l}{\partial \gamma^2} = \sum_2 \frac{C(B_2)}{1-\varrho_2^2} z_i z_i' + \sum_3 \frac{C(B_3)}{1-\varrho_3^2} z_i z_i' \quad (7.54)$$

$$\frac{\partial^2 l}{\partial \gamma \partial \beta_2'} = -\sum_2 C(B_2) \frac{1}{\sigma_2} \frac{\varrho_2}{1-\varrho_2^2} \mathbf{Z} \mathbf{X}' \quad (7.55)$$

$$\frac{\partial^2 l}{\partial \gamma \partial \sigma_2} = -\sum_2 \frac{\varrho_2 u_2}{\sigma_2^2 (1-\varrho_2^2)} C(B_2) \mathbf{Z} \quad (7.56)$$

$$\frac{\partial^2 l}{\partial \gamma \partial \varrho_2} = \sum_2 \left[ C(B_2) \frac{\frac{u_2}{\sigma_2} \varrho_2 \mathbf{Z}'\gamma}{(1-\varrho_2^2)^2} - \lambda(B_2) \frac{\varrho_2}{(1-\varrho_2^2)^{\frac{3}{2}}} \right] \mathbf{Z} \quad (7.57)$$

$$\frac{\partial^2 l}{\partial \gamma \partial \beta_3'} = -\sum_3 C(B_3) \frac{1}{\sigma_3} \frac{\varrho_3}{1-\varrho_3^2} \mathbf{Z} \mathbf{X}' \quad (7.58)$$

$$\frac{\partial^2 l}{\partial \gamma \partial \sigma_3} = -\sum_3 \frac{\varrho_3 u_3}{\sigma_3^2 (1-\varrho_3^2)} C(B_3) \mathbf{Z} \quad (7.59)$$

$$\frac{\partial^2 l}{\partial \boldsymbol{\gamma} \partial \varrho_3} = \sum_3 \left[ C(B_3) \frac{u_3}{\sigma_3} \frac{\varrho_3 \mathbf{Z}' \boldsymbol{\gamma}}{(1 - \varrho_3^2)^2} + \lambda(B_3) \frac{\varrho_3}{(1 - \varrho_3^2)^{\frac{3}{2}}} \right] \mathbf{Z} \quad (7.60)$$

$$\frac{\partial^2 l}{\partial \boldsymbol{\beta}_2 \partial \boldsymbol{\beta}'_2} = \sum_2 \frac{1}{\sigma_2^2} \left[ \frac{\varrho_2^2}{1 - \varrho_2^2} C(B_2) - 1 \right] \mathbf{X} \mathbf{X}' \quad (7.61)$$

$$\frac{\partial^2 l}{\partial \boldsymbol{\beta}_2 \partial \sigma_2} = \sum_2 \left[ C(B_2) \frac{u_2}{\sigma_2^3} \frac{\varrho_2^2}{1 - \varrho_2^2} - \frac{\lambda(B_2)}{\sigma_2^2} \frac{\varrho_2}{\sqrt{1 - \varrho_2^2}} - 2 \frac{u_2}{\sigma_2^3} \right] \mathbf{X} \quad (7.62)$$

$$\frac{\partial^2 l}{\partial \boldsymbol{\beta}_2 \partial \varrho_2} = \sum_2 \left[ -C(B_2) \frac{\frac{u_2}{\sigma_2} + \varrho_2 \mathbf{Z}' \boldsymbol{\gamma}}{(1 - \varrho_2^2)^2} \frac{\varrho_2}{\sigma_2} + \frac{\lambda(B_2)}{\sigma_2} \frac{1}{(1 - \varrho_2^2)^{\frac{3}{2}}} \right] \mathbf{X} \quad (7.63)$$

$$\frac{\partial^2 l}{\partial \boldsymbol{\beta}_2 \partial \boldsymbol{\beta}_3} = 0 \quad (7.64)$$

$$\frac{\partial^2 l}{\partial \boldsymbol{\beta}_2 \partial \sigma_3} = 0 \quad (7.65)$$

$$\frac{\partial^2 l}{\partial \boldsymbol{\beta}_2 \partial \varrho_3} = 0 \quad (7.66)$$

$$\begin{aligned} \frac{\partial^2 l}{\partial \sigma_2^2} &= \sum_2 \left[ \frac{1}{\sigma_2^2} - 3 \frac{u_2^2}{\sigma_2^4} + \frac{u_2}{\sigma_2^4} \frac{\varrho_2^2}{1 - \varrho_2^2} C(B_2) \right] - \\ &- 2 \sum_2 \lambda(B_2) \frac{u_2}{\sigma_2^3} \frac{\varrho_2}{\sqrt{1 - \varrho_2^2}} \end{aligned} \quad (7.67)$$

$$\frac{\partial^2 l}{\partial \sigma_2 \partial \varrho_2} = \frac{1}{(1 - \varrho_2^2)^{\frac{3}{2}}} \sum_2 \frac{u_2}{\sigma_2^2} \left[ -C(B_2) \frac{\varrho_2 \left( \frac{u_2}{\sigma_2} + \varrho_2 \mathbf{Z}' \boldsymbol{\gamma} \right)}{\sqrt{1 - \varrho_2^2}} + \lambda(B_2) \right] \quad (7.68)$$

$$\frac{\partial^2 l}{\partial \sigma_2 \partial \boldsymbol{\beta}_3} = 0 \quad (7.69)$$

$$\frac{\partial^2 l}{\partial \sigma_2 \partial \sigma_3} = 0 \quad (7.70)$$

$$\frac{\partial^2 l}{\partial \sigma_2 \partial \varrho_3} = 0 \quad (7.71)$$

$$\begin{aligned} \frac{\partial^2 l}{\partial \varrho_2^2} &= \sum_2 C(B_2) \left[ \frac{\frac{u_2}{\sigma_2} + \varrho_2 \mathbf{Z}' \boldsymbol{\gamma}}{(1 - \varrho_2^2)^{\frac{3}{2}}} \right]^2 - \\ &- \sum_2 \frac{\phi(B_2)}{\Phi(B_2)} \frac{\mathbf{Z}' \boldsymbol{\gamma} (1 + 2\varrho_2^2) + 3\varrho_2 \frac{u_2}{\sigma_2}}{(1 - \varrho_2^2)^{\frac{3}{2}}} \end{aligned} \quad (7.72)$$

$$\frac{\partial^2 l}{\partial \varrho_2 \partial \boldsymbol{\beta}_3} = 0 \quad (7.73)$$

$$\frac{\partial^2 l}{\partial \varrho_2 \partial \sigma_3} = 0 \quad (7.74)$$

$$\frac{\partial^2 l}{\partial \varrho_2 \partial \varrho_3} = 0 \quad (7.75)$$

$$\frac{\partial^2 l}{\partial \beta_3 \partial \beta_3'} = \sum_3 \frac{1}{\sigma_3^2} \left[ \frac{\varrho_3^2}{1 - \varrho_3^2} C(B_3) - 1 \right] \mathbf{X} \mathbf{X}' \quad (7.76)$$

$$\frac{\partial^2 l}{\partial \beta_3 \partial \sigma_3} = \sum_3 \left[ C(B_3) \frac{\varrho_3^2}{\sigma_3^3} \frac{u_3}{1 - \varrho_3^2} + \frac{\varrho_3}{\sigma_3^2} \frac{\lambda(B_3)}{\sqrt{1 - \varrho_3^2}} - 2 \frac{u_3}{\sigma_3^3} \right] \mathbf{X} \quad (7.77)$$

$$\frac{\partial^2 l}{\partial \beta_3 \partial \varrho_3} = \sum_3 \left[ -C(B_3) \frac{\frac{u_3}{\sigma_3} + \varrho_3 \mathbf{Z}' \boldsymbol{\gamma}}{(1 - \varrho_3^2)^2} \frac{\varrho_3}{\sigma_3} - \frac{\lambda(B_3)}{\sigma_3} \frac{1}{(1 - \varrho_3^2)^{\frac{3}{2}}} \right] \mathbf{X} \quad (7.78)$$

$$\begin{aligned} \frac{\partial^2 l}{\partial \sigma_3^2} &= \sum_3 \left[ \frac{1}{\sigma_3^2} - 3 \frac{u_3^2}{\sigma_3^4} + 2 \lambda(B_3) \frac{y_3 - \mathbf{X}' \boldsymbol{\beta}_3}{\sqrt{1 - \varrho_3^2}} \frac{\varrho_3}{\sigma_3^3} \right] + \\ &+ \sum_3 \frac{\varrho_3^2}{\sigma_3^4} \frac{u_3^2}{1 - \varrho_3^2} C(B_3) \end{aligned} \quad (7.79)$$

$$\frac{\partial^2 l}{\partial \sigma_3 \partial \varrho_3} = -\frac{1}{(1 - \varrho_3^2)^{\frac{3}{2}}} \sum_3 \frac{u_3}{\sigma_3^2} \left[ C(B_3) \frac{\varrho_3 \left( \frac{u_3}{\sigma_3} + \varrho_3 \mathbf{Z}' \boldsymbol{\gamma} \right)}{\sqrt{1 - \varrho_3^2}} + \lambda(B_3) \right] \quad (7.80)$$

$$\begin{aligned} \frac{\partial^2 l}{\partial \varrho_3^2} &= \sum_3 C(B_3) \left[ \frac{\frac{1}{\sigma_3} u_3 + \varrho_3 \mathbf{Z}' \boldsymbol{\gamma}}{(1 - \varrho_3^2)^{\frac{3}{2}}} \right]^2 + \\ &+ \sum_3 \lambda(B_3) \frac{\mathbf{Z}' \boldsymbol{\gamma} (1 + 2\varrho_3^2) + 3\varrho_3 \frac{1}{\sigma_3} u_3}{(1 - \varrho_3^2)^{\frac{7}{2}}} \end{aligned} \quad (7.81)$$

Siin on tähistatud

$$B_2 = \frac{\mathbf{Z}' \boldsymbol{\gamma} + \frac{\varrho_2}{\sigma_2} (y_2 - \mathbf{X}' \boldsymbol{\beta}_2)}{\sqrt{1 - \varrho_2^2}} \quad (7.82)$$

$$B_3 = \frac{\mathbf{Z}' \boldsymbol{\gamma} + \frac{\varrho_3}{\sigma_3} (y_3 - \mathbf{X}' \boldsymbol{\beta}_3)}{\sqrt{1 - \varrho_3^2}} \quad (7.83)$$

$$\lambda(B) = \frac{\phi(B)}{\Phi(B)} \quad (7.84)$$

$$u_2 = y_2 - \mathbf{X}' \boldsymbol{\beta}_2 \quad (7.85)$$

$$u_3 = y_3 - \mathbf{X}' \boldsymbol{\beta}_3 \quad (7.86)$$

$$C(B) = -\frac{\Phi(B)\phi(B)B + \phi(B)^2}{\Phi(B)^2} \quad (7.87)$$

## 7.4 Kestusmudelid

Tähistused:

$\tau$  kestus, algseisundis viibitud aeg

$t$  kalendriaeg

### 7.4.1 Kaplan-Meieri hinnang

**KM hinnang diskreetses ajas** Olgu perioodil  $j$   $r_j$  inimest "riski hulgas", s.t.  $r_j$  inimest võiksid põhimõtteliselt seisundist lahkuda. Lahkugu tegelikult  $n_j$  inimest,  $r_j - n_j$  jäävad edasi algseisundisse. KM hinnang hasardile on seega

$$\hat{h} = \frac{n_j}{r_j} \quad (7.88)$$

$$\widehat{\text{Var}} \hat{h}_j = \frac{\hat{h}_j(1 - \hat{h}_j)}{r_j}. \quad (7.89)$$

Kui periood  $j$  on  $k$  kuu pikkune, siis (keskmise) ühe kuu spetsiifilise hasardi saab

$$\hat{\vartheta}_j = 1 - (1 - \hat{h}_j)^{1/k} \quad (7.90)$$

$$\widehat{\text{Var}} \hat{\vartheta}_j = \frac{\text{Var} \hat{h}_j}{\left[ k(1 - \hat{h}_j)^{1-1/k} \right]^2} \quad (7.91)$$

**KM hinnang pidevas ajas** Lahkugu aja  $t$  jooksul  $r$  algseisundis olnud inimesest  $n$ . Keskmise hasart ajaühikus on

$$\hat{\vartheta} = -\frac{1}{t} \log(1 - n/r) \quad (7.92)$$

$$\widehat{\text{Var}} \hat{\vartheta} = \frac{1}{t^2} \frac{n/r}{r - n}. \quad (7.93)$$

### 7.4.2 Multiplikatiivne mittevaadeldav heterogeensus

Eeldame et hasart avaldub

$$\vartheta(\tau|x, v) = \lambda(\tau|x)v \quad (7.94)$$

kus  $v$  on mingi kindla jaotusega mittevaadeldav juhuslik suurus. Nüüd  $v$  keskvärtus algseisundisse jääjatel sõltub ajast:

$$\mathbb{E}(v|T \geq \tau) = -\frac{\mathcal{L}'[z(\tau|x)]}{\mathcal{L}[z(\tau|x)]} \quad (7.95)$$

kus on integreeritud hasart  $v$ -d arvestamata:

$$z(\tau|x) = \int_0^\tau \lambda(s|x) ds \quad (7.96)$$

Kui  $v$  on algseisundisse sissevoolus ühikdispersiooniga gammajaotus parameetriga  $\alpha$ , siis

$$\mathbb{E}(v|T \geq \tau) = \frac{\alpha}{z(\tau|x) + \alpha^{1/2}}. \quad (7.97)$$

### 7.4.3 Tükati konstantne põhihasart ja diskreetne mittevaadeldav heterogeensus ning pidev aeg

**Sõltumatud vaatlused** Eeldatakse, et hasart on konstantne iga  $M$  ajavahemiku sees, erinevatel ajavahemikel võib ta aga olla erinev. Olgu hasart kirjeldatud vektoriga  $\lambda$ , kusjuures ajavahemiku  $j$  põhihasart olgu  $e^{\lambda_j}$ . Mittevaadeldav heterogeensus on diskreetse jaotusega:

$$v = \begin{cases} v_h \equiv 1, & \text{tõenäosusega } p_h, \\ v_l, & \text{tõenäosusega } p_l = 1 - p_h. \end{cases} \quad (7.98)$$

Sobiv on parametrizeerida

$$\begin{aligned} v_1 &= e^{\tilde{v}_1} & p_1 &= \Lambda(\tilde{p}_1) \\ \dots & & \text{ja} & \dots \\ v_K &= 1 & p_K &= 1 - \sum^{K-1} p_k, \end{aligned} \quad (7.99)$$

Kus  $\Lambda(\cdot)$  on logistile jaotusfunktsioon ja  $v_k \in \mathfrak{X}$  ning  $p_k \in \mathfrak{X}$ .

Olgu  $m_i$  ajaperiood, mille jooksul inimene lahkub uuritavast seisundist ja tsenseerimist kirjeldagu  $\delta_i = 0$  kui vaatlus on tsenseeritud ja 1 kui tsenseerimata ning  $\mu_i = e^{\gamma x_i}$  olgu hasardi inimesest sõltuv osa.  $T_{ij}$  olgu teadaolev (võimalik et tsenseeritud) aeg, mis inimene  $i$  veetis uuritavas seisundis ajaperioodi  $j$  jooksul. Vektor  $T_i$  on  $M$ -vektor, mille komponendid on  $T_{ij}$  ning vektor  $d_i$  on vektor, mille  $j$ -s komponent on 1, kui inimene lahkus uuritavast seisundist ajaperioodil  $j$ . Muud komponendid on nullid.

Sel juhul inimese  $i$  laiklihuud avaldub:

$$\mathcal{L}_i = p_l \mathcal{L}_{li} + p_h \mathcal{L}_{hi} = p_l (v_l \mu_i e^{\lambda_{m_i}})^{\delta_i} e^{-z_{li}} + p_h (\mu_i e^{\lambda_{m_i}})^{\delta_i} e^{-z_{hi}}, \quad (7.100)$$

kus

$$z_{li} = v_l \mu_i \sum_{j=1}^{M-1} e^{\lambda_j} T_{ij} \quad \text{ja} \quad z_{hi} = \mu_i \sum_{j=1}^{M-1} e^{\lambda_j} T_{ij} \quad (7.101)$$

on integreeritud hasart. Log-laiklihuudi gradient avaldub:

$$\frac{\partial \ell_i}{\partial v_l} = \frac{p_l \mathcal{L}_{li}}{\mathcal{L}_i} \left[ \frac{\delta_i}{v_l} - z_{hi} \right] \quad (7.102)$$

$$\frac{\partial \ell_i}{\partial p_l} = \frac{1}{\mathcal{L}_i} [\mathcal{L}_{li} - \mathcal{L}_{hi}] \quad (7.103)$$

$$\frac{\partial \ell_i}{\partial \lambda} = p_l \frac{\mathcal{L}_{li}}{\mathcal{L}_i} (\delta_i d_i - v_l \mu_i T_i) + p_h \frac{\mathcal{L}_{hi}}{\mathcal{L}_i} (\delta_i d_i - \mu_i T_i) \quad (7.104)$$

$$\frac{\partial \ell_i}{\partial \gamma} = \frac{x_i}{\mathcal{L}_i} [p_l \mathcal{L}_l (\delta_i - z_{li}) + p_h \mathcal{L}_h (\delta_i - z_{hi})] \quad (7.105)$$

$$(7.106)$$

ja hessi maatriks:

$$\frac{\partial^2 \ell_i}{\partial v_l^2} = \frac{\mathcal{L}_{li}}{\mathcal{L}_i} \left[ \left( \frac{\delta_i}{v_l} - z_{hi} \right)^2 \left( 1 - p_l \frac{\mathcal{L}_{li}}{\mathcal{L}_i} \right) - \frac{\delta_i}{v_l^2} \right] \quad (7.107)$$

$$\frac{\partial^2 \ell_i}{\partial p_l^2} = -\left(\frac{\mathcal{L}_{li}}{\mathcal{L}_i} - \frac{\mathcal{L}_{hi}}{\mathcal{L}_i}\right)^2 \quad (7.108)$$

$$\frac{\partial^2 \ell_i}{\partial v_l \partial p_l} = \frac{\mathcal{L}_{li}}{\mathcal{L}_i} \left(\frac{\delta_i}{v_l} - z\right) \left(1 - p_l \frac{\mathcal{L}_{li} - \mathcal{L}_{hi}}{\mathcal{L}_i}\right) \quad (7.109)$$

$$\begin{aligned} \frac{\partial^2 \ell_i}{\partial \lambda \partial \lambda'} &= \frac{p_l}{\mathcal{L}_i} \left[ \frac{\partial \mathcal{L}_{li}}{\partial \lambda} (\delta_i \mathbf{d}_i - v_l \mu_i \mathbf{T}_i) - \mathcal{L}_{li} \text{diag}(v_l \mu_i \mathbf{e}^\lambda * \mathbf{T}_i) \right] + \\ &+ \frac{p_h}{\mathcal{L}_i} \left[ \frac{\partial \mathcal{L}_{hi}}{\partial \lambda} (\delta_i \mathbf{d}_i - \mu_i \mathbf{T}_i) - \mathcal{L}_{hi} \text{diag}(\mu_i \mathbf{e}^\lambda * \mathbf{T}_i) \right] - \\ &- \frac{1}{\mathcal{L}_i^2} \left(\frac{\partial \mathcal{L}_i}{\partial \lambda}\right)^2 \end{aligned} \quad (7.110)$$

$$\begin{aligned} \frac{\partial^2 \ell_i}{\partial \gamma \partial \gamma'} &= p_l \frac{\mathcal{L}_{li}}{\mathcal{L}_i} [(\delta_i - z_{li})^2 - z_{li}] \mathbf{x}_i \mathbf{x}_i' + p_h \frac{\mathcal{L}_{hi}}{\mathcal{L}_i} [(\delta_i - z_{hi})^2 - z_{hi}] \mathbf{x}_i \mathbf{x}_i' - \\ &- \frac{\partial \mathcal{L}_i}{\partial \gamma} \frac{\partial \mathcal{L}_i}{\partial \gamma'} \end{aligned} \quad (7.111)$$

$$\begin{aligned} \frac{\partial^2 \ell_i}{\partial \lambda \gamma'} &= [(\delta_i - z_{li})(\delta_i \mathbf{d}_i - z_{li}) - z_{li}] \frac{\mathcal{L}_{li}}{\mathcal{L}_i} \mathbf{x}_i' p_l + \\ &+ [(\delta_i - z_{hi})(\delta_i \mathbf{d}_i - z_{hi}) - z_{hi}] \frac{\mathcal{L}_{hi}}{\mathcal{L}_i} \mathbf{x}_i' p_h - \frac{1}{\mathcal{L}_i} \frac{\partial \mathcal{L}_i}{\partial \lambda} \frac{\partial \mathcal{L}_i}{\partial \gamma'} \end{aligned} \quad (7.112)$$

$$\begin{aligned} \frac{\partial^2 \ell_i}{\partial \lambda \partial v_l} &= p_l \frac{\mathcal{L}_{li}}{\mathcal{L}_i} \left(\frac{\delta_i}{v_l} - z_{hi}\right) \left(\delta_i \mathbf{d}_i - v_l \mu_i \mathbf{e}^\lambda * \mathbf{T}_i - \frac{1}{\mathcal{L}_i} \frac{\partial \mathcal{L}_i}{\partial \lambda}\right) - \\ &- p_l \frac{\mathcal{L}_{li}}{\mathcal{L}_i} \mu_i \mathbf{e}^\lambda * \mathbf{T}_i \end{aligned} \quad (7.113)$$

$$\begin{aligned} \frac{\partial^2 \ell_i}{\partial \lambda \partial p_l} &= \frac{\mathcal{L}_{li}}{\mathcal{L}_i} (\delta_i \mathbf{d}_i - v_l \mu_i \mathbf{e}^\lambda * \mathbf{T}_i) - \frac{\mathcal{L}_{hi}}{\mathcal{L}_i} (\delta_i \mathbf{d}_i - \mu_i \mathbf{e}^\lambda * \mathbf{T}_i) - \\ &- \frac{\mathcal{L}_{li} - \mathcal{L}_{hi}}{\mathcal{L}_i} \frac{1}{\mathcal{L}_i} \frac{\partial \mathcal{L}_i}{\partial \lambda} \end{aligned} \quad (7.114)$$

$$\begin{aligned} \frac{\partial^2 \ell_i}{\partial \gamma \partial v_l} &= p_l \frac{\mathcal{L}_{li}}{\mathcal{L}_i} \left(\frac{\delta_i}{v_l} - z_{hi}\right) (\delta_i - z_{li}) \mathbf{x}_i - \\ &- p_l \frac{\mathcal{L}_{li}}{\mathcal{L}_i} \left(\frac{\delta_i}{v_l} - z_{hi}\right) \frac{1}{\mathcal{L}_i} \frac{\partial \mathcal{L}_i}{\partial \gamma} \mathbf{x}_i - p_l \frac{\mathcal{L}_{li}}{\mathcal{L}_i} z_i \mathbf{x}_i \end{aligned} \quad (7.115)$$

$$\frac{\partial^2 \ell_i}{\partial \gamma \partial p_l} = \left[ \frac{\mathcal{L}_{li}}{\mathcal{L}_i} (\delta_i - z_{li}) - \frac{\mathcal{L}_{hi}}{\mathcal{L}_i} (\delta_i - z_{hi}) \right] \mathbf{x}_i - \frac{\mathcal{L}_{li} - \mathcal{L}_{hi}}{\mathcal{L}_i} \frac{1}{\mathcal{L}_i} \frac{\partial \mathcal{L}_i}{\partial \gamma} \quad (7.116)$$

Eelnevas tähendab  $\{\mathbf{e}^\lambda\}_i = \mathbf{e}^{\lambda_i}$ , \* vektorite elementide kaupa korrutamist ( $\{\mathbf{a} * \mathbf{b}\}_i = a_i b_i$ ) ning  $\text{diag } \mathbf{a}$  on maatriks, mille peadiagonaalil on vektor  $\mathbf{a}$  ja mujal nullid.

**Indiviidi-spetsiifiline heterogeensus** Eeldame nii nagu eespool et hasart on konstantne iga  $M$  ajavahemiku sees, erinevatel ajavahemikel võib ta aga olla erinev. Avaldugu hasart

$$\theta(t|\mathbf{x}, v) = v e^{\lambda(t)} e^{\gamma x'_i}. \quad (7.117)$$

Mittevaadeldav heterogeensus olgu diskreetse jaotusega  $v \in \{v_1, v_2, \dots, v_K\}$  ja tõenäosusega vastavalt  $p_1, p_2, \dots, p_K$ . Olgu inimese mittevaadeldav tunnus  $v$  ajas muutumatu, vaadeldav tunnus aga võib muutuda. Inimese  $i$  spelli  $j$  osa laiklihuudi funktsioonis on siis:

$$\mathcal{L}_{ij}(\cdot|v) = v \theta(t_{ij}|\mathbf{x}_{ij})^{\delta_{ij}} S(t_{ij}|\mathbf{x}_{ij}, v). \quad (7.118)$$

Siin  $t_{ij}$  on spelli vaadeldud kestus,  $\delta_{ij}$  on mitte-tsenseerituse indikaator ja  $\mathbf{x}_{ij}$  on inimese  $i$  vaadeldavad isikutunnused spelli  $j$  ajal.  $\mathcal{L}_{ij}(\cdot|v)$  on analoogne indiviidi-spetsiifilise laiklihuudiga sõltumatute vaatluste juhul.

Olgu inimese  $i$  kohta  $N_i$  vaadeldud spelli. Inimese  $i$  osa laiklihuudis avaldub siis

$$\mathcal{L}_i(\cdot|v) = \prod_{j=1}^{N_i} \mathcal{L}_{ij}(\cdot|v). \quad (7.119)$$

Vaadeldav laiklihuud avaldub:

$$\mathcal{L}_i(\cdot) = \sum_{k=1}^K p_k \mathcal{L}_i(\cdot|v_k). \quad (7.120)$$

Log-laiklihuudi gradient avaldub:

$$\frac{\partial \ell_i}{\partial v_k} = \frac{p_k}{\mathcal{L}_i(\cdot)} \sum_j \frac{\mathcal{L}_i(\cdot|v_k)}{\mathcal{L}_{ij}(\cdot|v_k)} \frac{\partial \mathcal{L}_{ij}(\cdot|v_k)}{\partial v_k} \quad (7.121)$$

$$\frac{\partial \ell_i}{\partial p_l} = \frac{\mathcal{L}_i(\cdot|v_k)}{\mathcal{L}_i(\cdot)} \quad (7.122)$$

$$\frac{\partial \ell_i}{\partial \lambda} = \frac{1}{\mathcal{L}_i(\cdot)} \sum_k p_k \sum_j \frac{\mathcal{L}_i(\cdot|v_k)}{\mathcal{L}_{ij}(\cdot|v_k)} \frac{\partial \mathcal{L}_{ij}(\cdot|v_k)}{\partial \lambda} \quad (7.123)$$

$$\frac{\partial \ell_i}{\partial \gamma} = \frac{1}{\mathcal{L}_i(\cdot)} \sum_k p_k \sum_j \frac{\mathcal{L}_i(\cdot|v_k)}{\mathcal{L}_{ij}(\cdot|v_k)} \frac{\partial \mathcal{L}_{ij}(\cdot|v_k)}{\partial \gamma}. \quad (7.124)$$

$\mathcal{L}_{ij}(\cdot|v)$  tuletised on analoogilised nagu sõltumatute vaatluste korral. Lisaks tuleb  $p$  järgi gradiendi võtmisel arvestada, et  $\sum p_k = 1$ .

#### 7.4.4 Intervallandmed

Intervallandmetega on tegemist siis kui on vaadeldav ainult fakt et sündmus (seisundite vahetamine, tsenseerimine) toimus mingis kestuse(aja)vahemikus (näiteks kuu või nädala jooksul). Intervallmudel sobib ka siis kui ei soovi hasarti täpselt spetsifitseerida.

Mudel: olgu kestus jagatud  $T+1$  vahemikuks:  $[0, t_1), [t_1, t_2), \dots, [t_{T-1}, t_T), [t_T, \infty)$ . Iga indiviidi  $i$  kohta olgu vaadeldav et millises vahemikus ta algsest seisundist lahkus, või et millises vahemikust vaatlus on tsenseeritud. Eeldame MPH mudelit nagu 7.4.3. osas:

$$\vartheta(\tau|\mathbf{x}, v) = \lambda(\tau)e^{\beta' \mathbf{x} v}. \quad (7.125)$$

Tõenäosus, et isik jääb kogu intervalli  $n$  jooksul algseisundisse avaldub

$$S_n(\mathbf{x}, v) = \exp(-vz_n(\mathbf{x})) = \exp(-v e^{\beta' \mathbf{x}} \int_{\tau_{n-1}}^{\tau_n} \lambda(s) ds). \quad (7.126)$$

Nüüd võib defineerida  $\tilde{\lambda}_n$ :

$$e^{\tilde{\lambda}_n} (t_n - t_{n-1}) \equiv \int_{\tau_{n-1}}^{\tau_n} \lambda(s) ds, \quad (7.127)$$

kus  $e^{\tilde{\lambda}_s}$  on keskmine põhihasart vahemikus  $s$  ja  $\tilde{\lambda}$  on lihtsalt mudeli parameeter. Seega põhihasart on spetsifitseeritud mitteparameetriselt. Vaatluse laiklihuud fikseeritud  $v$  korral avaldub nüüd

$$\mathcal{L}(n|\mathbf{x}, v) = \left(1 - e^{-vz_n(\mathbf{x})}\right)^\delta \prod_{m=1}^{n-1} e^{-vz_m(\mathbf{x})}, \quad (7.128)$$

kus  $\delta = 1$  tähendab et vaatlus pole tsenseeritud. Kogu vaatluse log-laiklihuud on

$$\ell(n|\mathbf{x}) = \log \left( \sum_{k=1}^K p_k \mathcal{L}(n|\mathbf{x}, v_k) \right). \quad (7.129)$$

$v$ -spetsiifilise laiklihuudi gradient avaldub

$$\frac{\partial}{\partial \beta} \mathcal{L}(n|\mathbf{x}, v) = v E_g(n|\mathbf{x}, v) \frac{\partial}{\partial \beta} z_n(\mathbf{x}) - v S_g(n|\mathbf{x}, v) \sum_{m=1}^{n-1} \frac{\partial}{\partial \beta} z_m(\mathbf{x}) \quad (7.130)$$

$$\frac{\partial}{\partial \tilde{\lambda}_s} \mathcal{L}(n|\mathbf{x}, v) = v E_g(n|\mathbf{x}, v) \frac{\partial}{\partial \tilde{\lambda}_s} z_n(\mathbf{x}) - v S_g(n|\mathbf{x}, v) \sum_{m=1}^{n-1} \frac{\partial}{\partial \tilde{\lambda}_s} z_m(\mathbf{x}) \quad (7.131)$$

$$\frac{\partial}{\partial v} \mathcal{L}(n|\mathbf{x}, v) = E_g(n|\mathbf{x}, v) z_n(\mathbf{x}) - S_g(n|\mathbf{x}, v) \sum_{m=1}^{n-1} z_m(\mathbf{x}) \quad (7.132)$$

kus

$$E_g(n|\mathbf{x}, v) = \delta e^{-vz_n(\mathbf{x})} \prod_{m=1}^{n-1} e^{-vz_m(\mathbf{x})} \quad (7.133)$$

$$S_g(n|\mathbf{x}, v) = \mathcal{L}(n|\mathbf{x}, v) = \left(1 - e^{-vz_n(\mathbf{x})}\right)^\delta \prod_{m=1}^{n-1} e^{-vz_m(\mathbf{x})} \quad (7.134)$$

on gradiendi seisundist lahkumise ja seisundis kestmise spetsiifilised osad ja z gradient avaldub

$$\frac{\partial}{\partial \boldsymbol{\beta}} z_n(\mathbf{x}) = z_n(\mathbf{x}) \mathbf{x}_n \quad (7.135)$$

$$\frac{\partial}{\partial \tilde{\lambda}_s} z_n(\mathbf{x}) = z_n(\mathbf{x}) \mathbb{1}(s = n). \quad (7.136)$$

Kogulaiklihuudi gradient on

$$\frac{\partial}{\partial \boldsymbol{\beta}} \ell(n|\mathbf{x}) = \frac{1}{\mathcal{L}(n|\mathbf{x})} \left( \sum_{k=1}^K p_k \frac{\partial}{\partial \boldsymbol{\beta}} \mathcal{L}(n|\mathbf{x}, v) \right) \quad (7.137)$$

$$\frac{\partial}{\partial \tilde{\lambda}_s} \ell(n|\mathbf{x}) = \frac{1}{\mathcal{L}(n|\mathbf{x})} \left( \sum_{k=1}^K p_k \frac{\partial}{\partial \tilde{\lambda}_s} \mathcal{L}(n|\mathbf{x}, v) \right) \quad (7.138)$$

$$\frac{\partial}{\partial v_k} \ell(n|\mathbf{x}) = \frac{1}{\mathcal{L}(n|\mathbf{x})} p_k \frac{\partial}{\partial v_k} \mathcal{L}(n|\mathbf{x}, v) \quad (7.139)$$

$$\frac{\partial}{\partial p_k} \ell(n|\mathbf{x}) = \frac{\mathcal{L}(n|\mathbf{x}, v_k)}{\mathcal{L}(n|\mathbf{x})} \quad (7.140)$$

**Mitu lõppseisundit** Mitme lõppseisundiga mudelid esitatakse sageli *kompiiting risk* kujul. Too tähendab et kujutatakse ette et kõikidesse lõppseisunditesse viivad sõltumatud Markovi protsessid, realiseerub too mille aeg on kõige lühem. Vajadusel võib tsenseerimist kujutada ühena lõppseisunditest.

Kui kõik muutujad on vaadeldavad, ei erine mitme lõppseisundiga juhtub olukorrast kui modelleerida üksikuid lõppseisundeid sõltumatult, teised seisundid oleksid siis nagu tsenseeritud. Kui mudelis on mittevaadeldav heterogeensus, on pilt ainult veidi keerulisem.

Olgu  $M$  võimalikku lõppseisundit ja  $m \in \{1, \dots, M\}$  olgu lõppseisundi näitaja. Olgu seisundisse  $m$  ülemineku hasart, sõltuvalt vaadeldavatest ja mittevaadeldavatest parameetritest,  $\vartheta^m(\tau|\mathbf{x}, v^m)$ . Mittevaadeldav heterogeensus olgu  $M$ -mõõtmelise diskreetse jaotusega:  $v^m \in \{v_1^m, \dots, v_{K^m}^m\}$  kusjuures  $\mathbf{v} = (v_{k^1}^1, \dots, v_{k^M}^M)$  esineb tõenäosusega  $p_{k^1 \dots k^M}$ . Eeldame et erinevate spellide jooksul on  $v$  konstantne,  $\mathbf{x}$  aga võib muutuda.

Inimese  $i$  spelli  $j$ , osa laiklihuudi funktsioonis siirde  $m$  järgi on nüüd:

$$\mathcal{L}_{ij}^m(\cdot|v^m) = \left[ \vartheta^m(\tau_{ij}|\mathbf{x}_{ij}) \right]^{\delta_{ij}^m} S^m(\tau_{ij}|\mathbf{x}_{ij}, v^m). \quad (7.141)$$

Siirdega seotud liige  $\left[ \vartheta^m(\tau_{ij}|\mathbf{x}_{ij}) \right]^{\delta_{ij}^m}$  ja püsimisega seotud liige  $S^m(\tau_{ij}|\mathbf{x}_{ij}, v^m)$  avalduvad nii nagu ühe spelli ja ühe lõppseisundi puhul (vaata näiteks 7.4.3 või 7.4.4). Siin  $\tau_{ij}$  on spelli vaadeldud kestus,  $\delta_{ij}^m$  on mitte-tsenseerituse indikaator  $m$ -seisundi mõttes (= 1 kui läks seisundisse  $m$  ja 0 kui es lähe) ja  $\mathbf{x}_{ij}$  on inimese  $i$  vaadeldavad isikutunnused spelli  $j$  ajal. Spellide kogulaiklihuud on:

$$\mathcal{L}_i(\cdot|\mathbf{x}, \mathbf{v}) = \prod_{m=1}^M \mathcal{L}_{ij}^m(\cdot|\mathbf{x}_{ij}, v^m). \quad (7.142)$$

Kompiiting risk mudel eeldab et siirded on sõltumatud (kui kontrollida  $\mathbf{x}$  ja  $\mathbf{v}$  suhtes), seega siis laiklihuudi korrutis siirete kaupa.

Olgu inimese  $i$  kohta  $N_i$  vaadeldud spelli. Inimese  $i$  osa laiklihuudis avaldub siis

$$\mathcal{L}_i(\cdot|\mathbf{x}, \mathbf{v}) = \prod_{j=1}^{N_i} \mathcal{L}_{ij}(\cdot|\mathbf{x}, \mathbf{v}). \quad (7.143)$$

ja vaadeldav laiklihuud avaldub:

$$\mathcal{L}_i(\cdot|\mathbf{x}) = \sum_{k^1=1}^{K^1} \cdots \sum_{k^M=1}^{K^M} p_{k^1 \dots k^M} \mathcal{L}_i(\cdot|\mathbf{x}, \mathbf{v}). \quad (7.144)$$

Laiklihuudi gradient avaldub:

$$\frac{\partial \mathcal{L}_i(\cdot|\mathbf{x})}{\partial \lambda^m} = \sum_{k^1=1}^{K^1} \cdots \sum_{k^M=1}^{K^M} p_{k^1 \dots k^M} \mathcal{L}_i(\cdot|\mathbf{x}, \mathbf{v}) \sum_{j=1}^{N_i} \left[ \frac{1}{\mathcal{L}_{ij}^m(\cdot|\mathbf{x}v_{k^m}^m)} \frac{\partial \mathcal{L}_{ij}^m(\cdot|\mathbf{x}v_{k^m}^m)}{\partial \lambda^m} \right] \quad (7.145)$$

$$\frac{\partial \mathcal{L}_i(\cdot|\mathbf{x})}{\partial \gamma^m} = \sum_{k^1=1}^{K^1} \cdots \sum_{k^M=1}^{K^M} p_{k^1 \dots k^M} \mathcal{L}_i(\cdot|\mathbf{x}, \mathbf{v}) \sum_{j=1}^{N_i} \left[ \frac{1}{\mathcal{L}_{ij}^m(\cdot|\mathbf{x}v_{k^m}^m)} \frac{\partial \mathcal{L}_{ij}^m(\cdot|\mathbf{x}v_{k^m}^m)}{\partial \gamma^m} \right] \quad (7.146)$$

$$\frac{\partial \mathcal{L}_i(\cdot|\mathbf{x})}{\partial v_k^m} = \sum_{k^1=1}^{K^1} \cdots \sum_{k^{m-1}=1}^{K^{m-1}} \sum_{k^{m+1}=1}^{K^{m+1}} \cdots \sum_{k^M=1}^{K^M} p_{k^1 \dots k^{m-1} k k^{m+1} \dots k^M}$$

$$\cdot \mathcal{L}_i(\cdot|\mathbf{x}, \mathbf{v}) \sum_{j=1}^{N_i} \left[ \frac{1}{\mathcal{L}_{ij}^m(\cdot|\mathbf{x}v_k^m)} \frac{\partial \mathcal{L}_{ij}^m(\cdot|\mathbf{x}v_k^m)}{\partial \gamma^m} \right] \quad (7.147)$$

$$\frac{\partial \mathcal{L}_i(\cdot|\mathbf{x})}{\partial p_{k^1 \dots k^M}} = \mathcal{L}_i(\cdot|\mathbf{x}, (v_{k^1}^1, \dots, v_{k^M}^M)) \quad (7.148)$$

$\mathcal{L}_{ij}^m(\cdot|\mathbf{x}, \mathbf{v})$  tuletised on nii nagu sõltumatute vaatluste korral.

#### 7.4.5 Parametriseerimine

**Logistic transition probability in discrete-time** In discrete time, it is useful to parameterise the destination-specific exit probabilities logistically as

$$p^m \equiv \Pr(\text{exit to destination } m) = \frac{e^{\lambda^m}}{1 + \sum_k e^{\lambda^k}}. \quad (7.149)$$

The probabilities are guaranteed to be positive and their sum to be less than one while  $\lambda$ -s may be unbounded. The components of the corresponding gradient transformation matrix are

$$\frac{\partial}{\partial \lambda^k} p^m = \mathbb{1}(k = m)p_m - p_k p_m \quad (7.150)$$

and the inverse transformation

$$l^m = \log \frac{p^m}{1 - \sum_k p^k}. \quad (7.151)$$

**Transition probability in continuous time** A good choice for parametrising the time-dependent part of the hazard is

$$\lambda = e^{\tilde{\lambda}} \quad (7.152)$$

The  $\lambda$  is now guaranteed to be positive.

**Diskreetne mittevaadeldav heterogeensus** Diskreetne mittevaadeldav heterogeensus, üks lõppseisund:  $v \in \{v_1, v_2, \dots, v_K\}$  ja vastavad tõenäosused  $p_1, p_2, \dots, p_K$ .

Kui laiklihuudi maksimeerimisel kasutada vastavaid parameetreid  $\tilde{v}$  ning  $\tilde{p}$  ( $\tilde{N}$  parameetrit) kuid laiklihuudi arvutamiseks nad normaalkujule ( $N$  parameetrit) teisendada, siis peab arvestama, et vastavad gradiendi komponendid teisenevad:

$$\begin{pmatrix} \frac{\partial}{\partial \tilde{v}} \ell_i \\ \frac{\partial}{\partial \tilde{p}} \ell_i \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial \tilde{v}} v & \frac{\partial}{\partial \tilde{v}} p \\ \frac{\partial}{\partial \tilde{p}} v & \frac{\partial}{\partial \tilde{p}} p \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial v} \ell_i \\ \frac{\partial}{\partial p} \ell_i \end{pmatrix} \equiv \mathbf{C} \begin{pmatrix} \frac{\partial}{\partial v} \ell_i \\ \frac{\partial}{\partial p} \ell_i \end{pmatrix} \quad (7.153)$$

Maatriksi  $\mathbf{C}$  ridade arv vastab algkuju parameetrite arvule  $\tilde{N}$  ning veergude arv normaalkuju parameetrite arvule  $N$  (s.h. lineaarselt sõltuvad  $p_H$  ja  $v_H$ ). Kovariatsioonimaatriksi  $p$  ning  $v$  sisaldav osa on vastavalt:

$$\Sigma = \mathbf{C}' \tilde{\Sigma} \mathbf{C} \quad (7.154)$$

**Heterogeensus** Tõenäosused on võimalik parametriseerida kui

$$p_k = \frac{e^{\tilde{p}_k}}{\sum_{k=1}^K e^{\tilde{p}_k}} \quad \text{ja} \quad \sum_{k=1}^K \tilde{p}_k = 0, \quad (7.155)$$

kus  $K$  on jaotuse toetuspunktide arv.

**Keskväertuse normeerimine**  $v$  keskväertuse võib normeerida üheks:

$$v_k = e^{\tilde{v}_k} \quad \text{ja} \quad \sum_{k=1}^K p_k v_k = 1. \quad (7.156)$$

Vastav pöördteisendus tõenäosuste tarvis on

$$\tilde{p}_k = \log p_k - \frac{\sum_{l=1}^K \log p_l}{K}. \quad (7.157)$$

**C** komponendid on:

$$\frac{\partial}{\partial \tilde{v}_l} v_k = \begin{cases} v_k & \text{kui } k = l < K \\ -v_l \frac{p_l}{p_K} & \text{kui } k = K \\ 0 & \text{muudel juhtudel} \end{cases} \quad (7.158)$$

$$\frac{\partial}{\partial \tilde{v}_l} p_k = 0 \quad (7.159)$$

$$\frac{\partial}{\partial \tilde{p}_l} v_k = \begin{cases} \frac{1}{p_K} [(1 - p_K) + p_K v_k + (1 - p_l) v_l] & \text{kui } k = K \\ 0 & \text{muudel juhtudel} \end{cases} \quad (7.160)$$

$$\frac{\partial p_k}{\partial \tilde{p}_l} = -p_k(p_l - p_K) + \mathbb{1}(k = l)p_K - \mathbb{1}(K = k)p_K \quad (7.161)$$

**$v$  komponendi normeerimine** Defineerime

$$v_K = 1. \quad (7.162)$$

Näib et nii on intervallandmete juures tulemused mõnevõrra stabiilsemad, samas on põhihasarti raskem tõlgendada.

**C** komponendid on

$$\frac{\partial}{\partial \tilde{v}_l} v_k = \begin{cases} v_k & \text{kui } k = l < K \\ 0 & \text{muudel juhtudel} \end{cases} \quad (7.163)$$

$$\frac{\partial}{\partial \tilde{v}_l} p_k = 0 \quad (7.164)$$

$$\frac{\partial}{\partial \tilde{p}_l} v_k = 0 \quad (7.165)$$

$$\frac{\partial p_k}{\partial \tilde{p}_l} = -\frac{e^{\tilde{p}_k} (e^{\tilde{p}_l} - e^{\tilde{p}_H})}{(\sum_{i=1}^H e^{\tilde{p}_i})^2} + \mathbb{1}(k = l) \frac{e^{\tilde{p}_k}}{\sum_{i=1}^H e^{\tilde{p}_i}} + \mathbb{1}(K = k) \frac{e^{\tilde{p}_k}}{\sum_{i=1}^K e^{\tilde{p}_i}} \quad (7.166)$$

**M lõppseisundit** (7.153) asemel võib nüüd kirjutada:

$$\begin{pmatrix} \frac{\partial}{\partial \tilde{v}^1} \ell_i \\ \frac{\partial}{\partial \tilde{v}^2} \ell_i \\ \dots \\ \frac{\partial}{\partial \tilde{v}^M} \ell_i \\ \frac{\partial}{\partial \tilde{p}} \ell_i \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial \tilde{v}^1} v^1 & \frac{\partial}{\partial \tilde{v}^1} v^2 & \dots & \frac{\partial}{\partial \tilde{v}^1} v^M & \frac{\partial}{\partial \tilde{v}^1} p \\ \frac{\partial}{\partial \tilde{v}^2} v^1 & \frac{\partial}{\partial \tilde{v}^2} v^2 & \dots & \frac{\partial}{\partial \tilde{v}^2} v^M & \frac{\partial}{\partial \tilde{v}^2} p \\ \dots & \dots & \ddots & \dots & \dots \\ \frac{\partial}{\partial \tilde{v}^M} v^1 & \frac{\partial}{\partial \tilde{v}^M} v^2 & \dots & \frac{\partial}{\partial \tilde{v}^M} v^M & \frac{\partial}{\partial \tilde{v}^M} p \\ \frac{\partial}{\partial \tilde{p}} v^1 & \frac{\partial}{\partial \tilde{p}} v^2 & \dots & \frac{\partial}{\partial \tilde{p}} v^M & \frac{\partial}{\partial \tilde{p}} p \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial v^1} \ell_i \\ \frac{\partial}{\partial v^2} \ell_i \\ \dots \\ \frac{\partial}{\partial v^M} \ell_i \\ \frac{\partial}{\partial p} \ell_i \end{pmatrix} \\
\equiv \mathbf{C} \begin{pmatrix} \frac{\partial}{\partial v^1} \ell_i \\ \frac{\partial}{\partial v^2} \ell_i \\ \dots \\ \frac{\partial}{\partial v^M} \ell_i \\ \frac{\partial}{\partial p} \ell_i \end{pmatrix}, \quad (7.167)$$

kus  $\mathbf{C}$  on  $\tilde{N} \times N$  maatriks.

Olgu  $p_{Dk}^m$  suuna  $m$  spetsiifiline tõenäosus, s.t. millise tõenäosusega esineb väärtus  $v_k^m$ . Tuletised võib nüüd avaldada kui

$$\frac{\partial}{\partial \tilde{p}} v^m = \frac{\partial}{\partial \tilde{p}} p \frac{\partial}{\partial p} p_D^m \frac{\partial}{\partial p_D^m} v^m \quad (7.168)$$

## 8 Algorithms

**Array indexing** Let  $M$  be a  $N$ -dimensional array and  $m[i_1, i_2, \dots, i_N]$  its element where  $i_j \in \{1, \dots, K_j\}$  and  $K_j$  is the size of dimension  $j$ . Let the array be stored in memory in FORTRAN-type, i.e. the running index for element  $m[i_1, i_2, \dots, i_N]$  is

$$j = i_1 + K_1(i_2 - 1) + K_2(i_3 - 1) + \dots + K_{N-1}(i_N - 1). \quad (8.1)$$

The array indices can be found from the running index  $j$  in the following way:

$$\begin{aligned} i_1 &= 1 + (j - 1) \bmod K_1 \\ i_2 &= 1 + [(j - 1) \bmod K_1] // K_2 \\ i_3 &= 1 + [(j - 1) \bmod (K_1 K_2)] // K_3 \\ i_4 &= 1 + [(j - 1) \bmod (K_1 K_2 K_3)] // K_4 \\ &\dots \end{aligned}$$

where  $//$  is the integer division.

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